A TYPE OF VARIATION ON NEWTON'S METHOD

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1. Introduction. Newton's Method of successive approximations to a root z^* of $\phi(z)=0$ consists in choosing a first approximation z_0 and then successively applying the formula

$$z_n = z_{n-1} - \phi(z_{n-1})/\phi'(z_{n-1}) \tag{1}$$

for $n=1, 2, \cdots$. Now if $\phi'(z^*)\neq 0$ and $\phi''(z^*)$ exists, and if z_0 is sufficiently near z^*, we have from (1), [1], and the fact that $\phi(z^*)=0$,

$$\begin{aligned}(z_n - z^*) &= (z_{n-1} - z^*) - \frac{(z_{n-1} - z^*)\phi'(z^*) + O[(z_{n-1} - z^*)^2]}{\phi'(z^*) + O[(z_{n-1} - z^*)]} \\ &= (z_{n-1} - z^*)\left\{1 - \frac{1 + O[(z_{n-1} - z^*)]}{1 + O[(z_{n-1} - z^*)]}\right\} = (z_{n-1} - z^*)O[(z_{n-1} - z^*)] \\ &= O[(z_{n-1} - z^*)^2].\end{aligned}$$

Frame [2] and Wall [3], independently, developed the formula

$$z_n = z_{n-1} - \frac{2\phi(z_{n-1})\phi'(z_{n-1})}{2[\phi'(z_{n-1})]^2 - \phi(z_{n-1})\phi''(z_{n-1})} \tag{2}$$

to provide still more rapid convergence of the z_n. We can show by analysis like that used above that, if $\phi'(z^*)\neq 0$, $\phi'''(z^*)$ exists, and z_0 is sufficiently near z^*, then $(z_n - z^*)=O[(z_{n-1}-z^*)^3]$, a fact pointed out by Wall for polynomial $\phi(z)$ [3, p. 94].

The order relations indicated in the preceding two paragraphs suggest the following problem: To find a function $f(z)$ associated with $\phi(z)$ such that, if z_0 is sufficiently near z^* and $z_n=f(z_{n-1})$ for $n=1, 2, \cdots$, then $z_n\to z^*$ with $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$ for $k=2, 3, \cdots$ in general. It is our purpose to discuss this problem. Except for geometric interpretations, our analysis may be assumed to take place in either the real or the complex field.

2. Solution by inversion of series. The efficacy of (1) is usually suggested in the real field by showing that z_n is the z-intercept of the line tangent to the curve $y=\phi(z)$ at the point (z_{n-1}, y_{n-1}), where $y_{n-1}=\phi(z_{n-1})$; and Frame and Wall define the z_n of (2) by approximating with (1) the appropriate z-intercept of the vertically opening parabola which osculates the curve at this point. The z-intercept of the horizontally opening parabola is more easily computed, and this parabola we generalize to osculating curves with the equations

$$z = \psi(y_{n-1}) + (y - y_{n-1})\psi'(y_{n-1}) + \cdots + (y - y_{n-1})^{k-1}\psi^{(k-1)}(y_{n-1})/(k-1)!,$$

where $z=\psi(y)$ is inverse to $y=\phi(z)$. We assume that $\phi'(z^*)\neq 0$ (or that $\phi^{(s)}(z^*)\neq 0$ for some s, in which case we may replace $\phi(z)$ throughout by $g(z)\equiv\phi(z)/\phi'(z)$,

for which $g(z^*)=0$ and $g'(z^*)\neq 0$ [4, p. 119]). Putting $y=0$ we obtain

$$(3)\quad z_n = \psi(y_{n-1}) - y_{n-1}\psi'(y_{n-1}) + \cdots + (-1)^{k-1}\frac{k-1}{y_{n-1}}\psi^{(k-1)}(y_{n-1})/(k-1)!,$$

or

$$(4)\quad \begin{aligned} z_n = z_{n-1} &- \phi(z_{n-1})/\phi'(z_{n-1}) - [\phi(z_{n-1})]^2\phi''(z_{n-1})/\{2[\phi'(z_{n-1})]^3\} + \cdots \\ &+ (-1)^{k-1}[\phi(z_{n-1})]^{k-1}\psi^{(k-1)}[\phi(z_{n-1})]/(k-1)!. \end{aligned}$$

For example, taking $k=3$, we have

$$(5)\quad z_n = z_{n-1} - \phi(z_{n-1})\{2[\phi'(z_{n-1})]^2 + \phi(z_{n-1})\phi''(z_{n-1})\}/\{2[\phi'(z_{n-1})]^3\}.$$

Now (4) solves our problem if $\phi^{(k)}(z^*)$ exists. For then $\psi^{(k)}(0)$ exists, and we have (as in the derivation of (8a) in §5 below)

$$(6)\quad \begin{aligned} z^* = \psi(y_{n-1}) &- y_{n-1}\psi'(y_{n-1}) + \cdots \\ &+ (-1)^{k-1}\frac{k-1}{y_{n-1}}\psi^{(k-1)}(y_{n-1})/(k-1)! + O(y_{n-1}^k) \end{aligned}$$

in which, since $(z_{n-1}-z^*)/y_{n-1}\to\psi'(0)$, we may replace $O(y_{n-1}^k)$ by $O[(z_{n-1}-z^*)^k]$; and comparison of (6) and (3) shows that $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$.

We shall not content ourselves with this solution, however, since formulas for the derivatives of the inverse function in terms of the direct function and its derivatives [5] are somewhat more involved than the expressions which we shall obtain for $f(z)$ in the following section. Indeed if tractable formulas for the derivatives of the inverse function were available† we could write a useful, immediate expression for z^* itself in the case of analytic $\phi(z)$, namely, the infinite series of which (4) with $n=1$ includes the first k terms.‡

3. Solution by the method of [4]. Perhaps the most direct attack upon our problem is to pursue the method of analysis of [4], by means of which we shall obtain, for regular $\phi(z)$ and $f(z)$, the general $f(z)$ for which $(z_n-z^*)=O[(z_{n-1}-z^*)^k]$ for $k=2, 3, \cdots$. Supposing then that $\phi(z)$ and $f(z)$ are regular at z^*, we assume (or bring about as before) the condition $\phi'(z^*)\neq 0$, and write

$$f(z) = a_0 + a_1(z-z^*) + a_2(z-z^*)^2 + \cdots .$$

Since

$$z_n \equiv f(z_{n-1}) \equiv a_0 + a_1(z_{n-1}-z^*) + a_2(z_{n-1}-z^*)^2 + \cdots$$

must tend to z^*, we see that $a_0=z^*$. Hence

$$f(z) - z^* = a_1(z-z^*) + a_2(z-z^*)^2 + \cdots ,$$

† Securing formulas for these derivatives is the problem of inversion of series. The classical results are due to Van Orstrand [5], who lists expressions for $\psi^{(\nu)}$ for $\nu=1, 2, \cdots, 13$. The referee notes that a paper by Reynolds [6] suggests an easily understood process for calculating these expressions.

‡ I am indebted to the referee for calling my attention to a paper by Corey [7] which notes this fact.

so that

$$z_n - z^* = a_1(z_{n-1} - z^*) + a_2(z_{n-1} - z^*)^2 + \cdots .$$

Since we require that $(z_n - z^*) = O[(z_{n-1} - z^*)^k]$ it follows that $a_\nu = 0$ for $\nu < k$. So

$$f(z) = z^* + a_k(z - z^*)^k + \cdots \equiv z - (z - z^*) + a_k(z - z^*)^k + \cdots . \tag{7}$$

Conversely, if $f(z)$ is of the form (7) with $k \geqq 2$ it is clear that $z_n \to z^*$ (provided that z_0 is sufficiently near z^*; this is assumed throughout) with $(z_n - z^*) = O[(z_{n-1} - z^*)^k]$.

It remains to approximate the powers of $(z - z^*)$ in terms of $\phi(z)$ and its derivatives. To this end we first expand $\phi(z)$ about the point z and evaluate at z^*:

$$\phi(z^*) = \phi(z) + (z^* - z)\phi'(z) + (z^* - z)^2\phi''(z)/2! + \cdots .$$

Taking account of the vanishing of $\phi(z^*)$ and rewriting, we obtain

$$0 = \phi(z) - (z - z^*)\phi'(z) + (z - z^*)^2\phi''(z)/2! - \cdots . \tag{8}$$

Multiplying (8) by the successive powers of $(z - z^*)$ and adjoining the resulting equations to a rearrangement of (8), we obtain the system (in which the argument z is omitted for brevity)

$$\begin{aligned}
-(z - z^*)\phi' + (z - z^*)^2\phi''/2! - (z - z^*)^3\phi'''/3! + \cdots &= -\phi; \\
(z - z^*)\phi - (z - z^*)^2\phi' + (z - z^*)^3\phi''/2! - \cdots &= 0; \\
(z - z^*)^2\phi - (z - z^*)^3\phi' + \cdots &= 0; \\
(z - z^*)^3\phi - \cdots &= 0; \\
\cdots &
\end{aligned} \tag{9}$$

We may now find an expression which shall differ from $(z - z^*)^s$ to the order of $(z - z^*)^{t+1}$ (s, t arbitrary, with $t \geqq s$) by merely transposing all terms beyond the tth in each of the equations (9) and eliminating all powers of $(z - z^*)$ except the sth from the left-hand sides of the first t of the resulting equations. So doing, we obtain

$$(z - z^*)^s = (-1)^{s+1}\Delta_{t,s}/\Delta_t + O[(z - z^*)^{t+1}],$$

where Δ_t is the symmetric determinant,

$$\begin{vmatrix}
\phi' & \phi''/2! & \cdots & \phi^{(t-2)}/(t-2)! & \phi^{(t-1)}/(t-1)! & \phi^{(t)}/t! \\
\phi & \phi' & \cdots & \phi^{(t-3)}/(t-3)! & \phi^{(t-2)}/(t-2)! & \phi^{(t-1)}/(t-1)! \\
0 & \phi & \cdots & \phi^{(t-4)}/(t-4)! & \phi^{(t-3)}/(t-3)! & \phi^{(t-2)}/(t-2)! \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \phi & \phi' & \phi''/2! \\
0 & 0 & \cdots & 0 & \phi & \phi'
\end{vmatrix}$$

which, since $\phi(z^*)=0$ and $\phi'(z^*)\neq 0$, does not vanish at z^*, and where $\Delta_{t,s}$ is the same determinant with the sth column replaced by a column whose first element is ϕ and whose others are all zero. We note that

$$\Delta_{t,1} = \phi\Delta_{t-1}, \qquad (\Delta_0 \equiv 1), \qquad \Delta_{t,t} = \phi^t,$$
$$\Delta_t = \Delta_{t-1}\phi' - \Delta_{t-2}\phi\phi''/2! + \Delta_{t-3}\phi^2\phi'''/3! - \cdots + (-1)^k\Delta_0\phi^{k-1}\phi^{(k)}/k!, \tag{10}$$

the last relation being obtained by expanding Δ_t by the first column and using induction.

Returning to (7) we now see that $f(z)$ as there represented is of the form

$$\begin{aligned} f(z) &= z - \Delta_{k-1,1}/\Delta_{k-1} - O[(z-z^*)^k] + a_k(z-z^*)^k + \cdots \\ &= z - \Delta_{k-1,1}/\Delta_{k-1} + b_k(z-z^*)^k + b_{k+1}(z-z^*)^{k+1} + \cdots \\ &= z - \Delta_{k-1,1}/\Delta_{k-1} + (z-z^*)^k[b_k + b_{k+1}(z-z^*) + \cdots]. \end{aligned}$$

Hence, because z^* is a simple root of $\phi(z)$, we have (h is to be read a function which is regular at z^*)

THEOREM 1. *A necessary and sufficient condition that $z_n \to z^*$ with $(z_n - z^*) = O[(z_{n-1}-z^*)^k]$ for $k \geqq 2$ is that*

$$f(z) = z - \Delta_{k-1,1}/\Delta_{k-1} + h\phi^k \equiv z - \phi\Delta_{k-2}/\Delta_{k-1} + h\phi^k.$$

Since from (7) we have

$$(z_n - z^*)/(z_{n-1} - z^*) = a_k(z_{n-1} - z^*)^{k-1} + a_{k+1}(z_{n-1} - z^*)^k + \cdots,$$

the z_n will ultimately oscillate about z^* in the real domain if and only if k (assumed $\geqq 2$) is odd. We therefore have

THEOREM 2. *A necessary and sufficient condition that $z_n \to z^*$ with $(z_n - z^*) = O[(z_{n-1}-z^*)^2]$ and in the real domain ultimately oscillate about z^* is that*

$$\begin{aligned} f(z) &= z - \Delta_{2l+1,1}/\Delta_{2l+1} - a\Delta_{2l+1,2l+1}/\Delta_{2l+1} + h\phi^{2l+2} \\ &\equiv z - \phi(\Delta_{2l} + a\phi^{2l})/\Delta_{2l+1} + h\phi^{2l+2}, \end{aligned}$$

where $a>0$ and l is some positive integer. Necessarily, $(z_n - z^*) = O[(z_{n-1}-z^*)^{2l+2}]$. (Compare [4, p. 121].)

4. Special cases. The earlier formulas. Taking $h=0$ and $k=2$ in the formula of Theorem 1 and noting that $\Delta_1 = \phi'$, we obtain $f(z) = z - \phi/\phi'$, the iterative function of Newton's Method. Taking $h=0$ and $k=3$ in the same formula and using (10) to find that $\Delta_2 = \phi'^2 - \phi\phi''/2$, we obtain $f(z) = z - 2\phi\phi'/(2\phi'^2 - \phi\phi'')$, which is the iterative function of Frame and Wall. Taking $k=3$ in the same formula but leaving h for the moment unspecified we obtain

$$f(z) = z - \frac{2\phi\phi'}{2\phi'^2 - \phi\phi''} + h\phi^3 = z - \frac{\phi}{\phi'} \Big/ \left(1 - \frac{\phi''\phi}{2\phi'^2}\right) + h\phi^3$$

$$= z - \frac{\phi}{\phi'}\left(1 + \frac{\phi\phi''}{2\phi'^2}\right) - \phi^3\left(\frac{\phi''^2}{4\phi'^5} + \cdots\right) + h\phi^3;$$

if we now take

$$h = \frac{\phi''^2}{4\phi'^5} + \cdots$$

we finally obtain $f(z) = z - \phi(2\phi'^2 + \phi\phi'')/(2\phi'^3)$, which is the iterative function of (5).

Taking $h=0$, $l=1$, and $a=1$ in the formula of Theorem 2 and using (10) to find that $\Delta_3 = \phi'^3 + \phi^2\phi'''/6 - \phi\phi'\phi''$, we obtain $f(z) = z - 3\phi(2\phi^2 + 2\phi'^2 - \phi\phi'')/(6\phi'^3 + \phi^2\phi''' - 6\phi\phi'\phi'')$, a particular iterative function for which $(z_n - z^*) = O[(z_{n-1} - z^*)^4]$ and, in the real domain, the z_n ultimately oscillate about z^*.

In the real domain the graphical interpretation (see [4, p. 114]; compare [3, Figure 2]) of the situation in which the z_n oscillate about z^* is simply that the curve $y = f(z)$ descends in a deleted neighborhood of the point (z^*, z^*), and the rapidity of convergence of the z_n in Theorem 1 corresponds to the high order of contact ($\geqq (k-1)$th) of that curve with its horizontal tangent at (z^*, z^*).

5. Conclusion. *Remarks.* If instead of the regularity of $\phi(z)$ at z^* we have merely the existence of $\phi^{(k)}(z^*)$ (or of $\phi^{(2l+2)}(z^*)$), and if we impose no regularity requirements on $f(z)$, then the conditions of Theorems 1 and 2 are sufficient for the purposes indicated. This follows from a slight modification of the preceding analysis, starting with the relation (which we shall at once derive)

$$0 = \phi(z) - (z - z^*)\phi'(z) + \cdots + (-1)^{k-1}(z - z^*)^{k-1}\phi^{(k-1)}(z)/(k-1)! + O[(z - z^*)^k] \tag{8a}$$

instead of (8).

To validate (8a) we note first that $\phi^{(k-1)}(z)$ exists in a neighborhood of z^*, define R by

$$0 = \sum_{\mu=0}^{k-1} (z^* - z)^\mu \phi^{(\mu)}(z)/\mu! + R, \tag{11}$$

expand the $\phi^{(\mu)}(z)$ about z^*, rewrite (11), and collect terms. Thus

$$R = -\sum_{\mu=0}^{k-1} (z^* - z)^\mu \sum_{\nu=0}^{k-1-\mu} (z - z^*)^\nu \phi^{(\mu+\nu)}(z^*)/(\mu!\nu!) + O[(z - z^*)^k]$$

$$= -\sum_{\lambda=0}^{k-1} \left\{ \sum_{\nu+\mu=\lambda} [\lambda!/(\mu!\nu!)](z^* - z)^\mu (z - z^*)^\nu \right\} \phi^{(\lambda)}(z^*)/\lambda! + O[(z - z^*)^k]$$

$$= -\sum_{\lambda=0}^{k-1} [(z^* - z) + (z - z^*)]^\lambda \phi^{(\lambda)}(z^*)/\lambda! + O[(z - z^*)^k]$$

$$= -\phi(z^*) + O[(z - z^*)^k] = O[(z - z^*)^k].$$

Hence (11) identifies itself with (8a) and the proof is complete.

Among the remaining questions of possible interest are the following:

(**A**) We have seen that z_1 as given by (4) tends to z^* as k becomes infinite if $\phi(z)$ is regular at z^* and z_0 is sufficiently near z^*. With what other choices of h than those which give us the partial sums (4) of the expansion of the inverse function for regular $\phi(z)$ can the same conclusion be asserted?

(**B**) Are there choices of h other than zero which might make $f(z)$ in some sense simpler? (Compare the derivation of (5) in §4 above.)

(**C**) Are there approximations to the powers of $(z-z^*)$ which are more readily calculable than those of §3?

Although nothing can be proved concerning an infinite process by finite computation, it might be interesting to apply one of our formulas to a numerical case. Let us then approximate $\sqrt{2}$ by taking $\phi=z^2-2$. Here, with $h=0$, $f(z)=[z\Delta_k-(z^2-2)\Delta_{k-1}]/\Delta_k$ and, since $\phi'=2z$, $\phi''=2$, and $\phi'''=\phi^{iv}=\cdots=0$, we have from (10) $\Delta_k=2z\Delta_{k-1}-(z^2-2)\Delta_{k-2}$. Starting with $\Delta_0=1$ and $\Delta_1\equiv\phi'=2z$, we obtain, successively, $\Delta_2=3z^2+2$, $\Delta_3=4z^3+8z$, $\Delta_4=5z^4+20z^2+4$, $\Delta_5=6z^5+40z^3+24z$, $\cdots$. Taking $k=5$ we have, after simplifications,

$$f(z)=\frac{(z^2+2)(z^4+28z^2+4)}{2z(z^2+6)(3z^2+2)};$$

with $z_0=1$, this yields $z_1=1.41429-$ and with $z_0=1.5$ it yields $z_1=1.414213564+$. Actually, $\sqrt{2}=1.414213562+$.

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EDUCATION VS. LEGISLATION

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It all started in the last part of the nineteenth century; however, the change was so gradual that almost no one fully appreciated the fact that mathematics was losing its place of importance as a secondary school subject and as a general requirement for the bachelor's degree until after the end of World War I. At least, mathematicians were not too well aware of the development. When, in the early twenties, the trend began to make itself felt, mathematicians proved that they were human—they followed the first impulse of a trapped soul—they began to fight. The battle raged in literature and on college campuses; the mathematicians condemned bitterly the professional educators, calling upon the shades of Euclid and Pythagoras to witness their noble stand.* The professional educator was prepared for this onslaught, having his ramparts protected by a battery of tests (the validity of which were unquestionable!), a philosophic concept or two (which were undebatable!), and last, but perhaps the strongest weapon, the ears of the powers that be. This was truly a formidable foe against which the mathematicians had arrayed themselves. The outcome was inevitable; the proponents of mathematics steadily lost ground. The fray continued for at least one decade before there was any evidence of a change of approach on the part of the mathematicians, and this change was by no means a general one. Mathematicians are not to be censored too severely for the tactics used during the twenties; some good was no doubt done—at least the question of the role of mathematics in the curriculum was kept open and not allowed to die in the hands of the victors, as was true of one field of study, the passing of which is mourned by many.

In the thirties some few mathematicians began to work with the educators in curriculum planning, in the evaluation of secondary mathematics in the light of modern concepts of education, in the development of better teaching practices, and in the production of more desirable texts. The effects of this change of approach in many instances were fine; however, favorable accomplishments were not general because so few mathematicians, in comparison with the magnitude of the task, participated. There was still a feeling of animosity between the two groups involved. Had the entire forces of the mathematical band been united in this effort of working with the professional educator rather than trying to force him to accept certain ideas, much more and lasting good would have been accomplished. At least a spirit of friendliness and comradeship in a mutual problem would have been developed, which would have been conducive to to further progress. Since all of this did not happen, a stalemate still exists generally.

When the deplorable conditions pertaining to the mathematical knowledge of the youth of our land were made public by the testing agencies of the armed forces, mathematicians said, "I told you so," and settled back expecting edu-

* The author, a bit young at the beginning, soon became of age and joined the troops

cators to make the necessary adjustments and produce the desired results. Since this has not occurred, the battle has started all over again. Let us recall that which took place from 1920 to 1940, and change our approach to the problem. Unless this is done, conditions will remain unchanged. The balance of this paper will deal with suggestions for a new approach. In particular, an approach is advocated that is built along educational lines rather than through the tactics of force and legislative action.

Before going into a discussion of those things which it is believed will tend to bring about the desired results, it is in order to make a few comments in general on the educational process. Since most of us are in the profession of educating people, we recognize two basic truths connected with the process. They are: (1) the slowness of attaining any noteworthy accomplishment and (2) the permanency and potency of those things which are learned.

One of the greatest failings of mankind is that of impatience. We are in a hurry; we want things done over night and corrections made immediately. Hence, during the twenties and thirties we tried and succeeded to some extent to have the number of units in mathematics required for graduation from high school increased. In those states where we were successful, it did not last long, for many reasons. There are several other examples of this type of thing which will not be mentioned here. Had we not been so perturbed and in such a hurry, a program of an educational nature would have produced by now those things which were desired. When thinking in terms of educating people to our way of believing, we must consider a period of ten, twenty, fifty and even one hundred years—at least, not an interval of one, two or three years.

Now are presented some ideas and suggestions concerning the proposed approach. For the most part, these ideas find expression in the present program in Louisiana, or are involved in the plans for the immediate future. We in Louisiana do not have all the answers even though we believe that we are making progress; therefore, we shall be most happy to receive additional suggestions from any who read this paper.

There are four mathematical organizations operating within Louisiana. They are: (1) The Louisiana-Mississippi Section of the Mathematical Association of America, (2) The Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics, (3) The Mathematics Section of the Louisiana College Conference, and (4) The Mathematics Section of the Louisiana Educational Association. Each of these groups meets annually. At least a part of every program deals with current problems in the field of mathematics, that is, instruction at all levels, certification of teachers, teacher training, teacher procurement, and the like.

Back about 1940, it was decided to form a standing committee composed of three members from each of these four organizations. At this time the militant spirit still ran high. Thus the committee operated as a pressure group. Its chief concern was over the fact that the college freshman was not prepared to do even average work in college mathematics. The answer of course, at that time,

was to get the number of units required for graduation from high school increased from the then existing one unit. As a pressure group, the committee was quite successful. After several conferences with the state superintendent and his staff, both in his office and at an invitational dinner or so, the requirements were raised to three units. No, it did not last long. Pressure was relaxed, mostly due to the war, and as usual with those things which are gained through pressure, they are lost when a period of relaxation takes place. It was just as well, for the college freshman had not improved. Now it is realized that merely making all high school students take three years of mathematics is not the answer. In fact, it is believed that a large per cent of high school students should not take the traditional courses at all. It would be fine for them to take some mathematics to add to their culture and to prepare them for various problems they will meet in their respective spheres of activity, but not the usual courses that were a part of the high school curriculum until more recent years.

The committee is no longer a pressure group. It is now older and wiser, and has become a consultant organization on the one hand and a public relations or educational agency on the other. The time is past when the committee goes to the state superintendent or members of his staff in a body making demands "in the name of common sense." Members of the committee now drop by the offices of these officials occasionally for a chat to offer their services and discuss mutual problems. Be it understood that all of this is sincere—the officials know it and feel it, for they are calling upon the mathematicians of the state for aid and advice. The committee, however, is not all "chit-chat," for it is continually collecting data and other information of importance to the problem at hand.

A few years ago the state department of education felt that the certification regulations needed to be revised and so proceeded to do it. Two outstanding events took place and two outstanding regulations were brought into the code in the process. The events were: (1) A member of the committee was made a member of the certification committee and (2) the committee itself was invited to appear before the certification group in order to make its recommendations. The modifications in regulations were: (1) The mathematical requirement necessary for certification in order to teach high school mathematics was increased from six to eighteen semester hours, and (2) six semester hours was made a part of the general educational requirement for a teaching certificate at any level. This six-semester hour requirement will be elaborated later.

President Conant of Harvard said a few years ago:*

> I am almost tempted to generalize that the more educated the person, the less his knowledge of secondary-school education. Certainly the lack of knowledge among the professors of arts and sciences in our colleges and universities is proverbial. And with lack of information goes lack of understanding and lack of sympathy. As a result, on more than one campus we have almost a state of civil war between those who profess a knowledge of education and those who profess a knowledge of subjects which constitute a modern educational curriculum.

* Conant, J. B., A Truce Among Educators, *Teachers College Record*, December, 1944, pp. 157–163.

> This academic war has been in a sense inevitable, as I propose to show by a brief résumé of history, but to my mind an armistice has been for some years overdue. And it is for such an armistice that I should like to put in a good word this afternoon (and I might remark parenthetically that it takes two to make an armistice quite as much as to make a quarrel). My belief in the need for the cessation of hostilities comes not only from my general tendency to favor pacific methods of handling academic controversy, but also because I am really worried about the present lay reaction to educational matters. I am distressed by both the vehemence and the ignorance with which views about education are expressed publicly and privately by many prominent people. Now we can hardly expect the public to have a very clear understanding about educational problems when education is a house warring against itself. Hence my plea this afternoon for a "cease firing" order.

This statement has been taken seriously in Louisiana. The feeling is that the mathematician still knows best what subject matter the prospective teacher needs to know; nevertheless, the thoughts of the educator are being respected. The result is that the two groups are working together on all the campuses of the state in order to bring about the best possible conditions, and the conditions are greatly improved. The ".. . . state of civil war . . . " of which President Conant spoke is ceasing to exist.

A new project of the committee, which is entering its second year, is a testing program. Each department of mathematics of the state is giving an achievement test to all new students. These tests are sent to a central location where they are processed and various statistical measures are taken. The results are being used in various ways to foster an improvement in secondary instruction and for remedial purposes. As they are used, the educational approach is kept in mind rather than the "we told you so" attitude.

The next venture, which is expected to take shape in 1950, is a mathematical institute of one week to ten days, and a series of week-end conferences. The institute, to be held at some desirable location, will be for all who desire to attend. The conferences are to be held on each of the campuses of the state and are primarily for those who are within driving range. Two colleges have already held such conferences with excellent results. The idea of this work is to give the elementary and secondary teachers some in-service training, inform them of present conditions, and excite them to greater activity.

Another topic which is presently being given consideration concerns graduate study. Traditionally, every student who enters a graduate school to study mathematics is trained as though he is to become a research man. This is desirable for the comparatively few who are going into research. However, every department gets a good many students who like mathematics and desire to teach it—even in high school, but who have just about "reached their level" especially when they get into courses for graduates only. These students usually will become discouraged and quit, or they will be asked to leave, or with a great amount of work from a patient professor they will struggle through to a master's degree. The members of this group, while not the best material for research mathematicians, would make excellent high school teachers—much better than the English and history majors who are forced to teach mathematics

in many schools. It is possible that we are missing a great opportunity to aid our field of study by not setting up a curriculum for this type of student and encourage him towards working in the secondary field. Many such curricula have been suggested during the past ten years or so and are to be found in the literature. One of the most recent is by the writer.*

The last topic to be discussed is a further consideration of the six semester-hours of mathematics in the general education program which is required for certification. Many of us have felt that this requirement was our golden opportunity. To be more specific, the students taking such a six-hour course were to be prospective teachers of the elementary schools and of all high school subjects, except mathematics and those sciences which require several courses of mathematics in their preparation. Thinking further, this group would contain prospective principals and superintendents. This was an opportunity, for the mathematician was going to have a personal contact with that group of teachers (teachers to be) who had been causing much of the trouble. These were the ones who disliked mathematics, could not see any use in it, thought it too difficult and so told the students. Even the prospective student of science and engineering was affected. What better chance could the mathematician have for correcting some of these evils?

When the six-semester hour rule became effective, it was thought by the professional educator that the traditional courses of algebra and trigonometry could be improved upon for this group of students. He thought further that the program should contain some arithmetic for the sake of the prospective elementary teacher. The mathematician was ready for this occasion because he had been conditioned by the current ideas which are spreading abroad as exemplified by such articles as those of Allendoerfer, Newsom and Ore.**

The course that is presently given aims at three ideas. In the first place an effort is made to instruct the student in the more useful phases of elementary mathematics. This begins with a good review of arithmetic, and continues with topics of algebra, geometry, trigonometry and analytics. Secondly, the student is made to realize the part that mathematics has played in the progress of civilization. This is done through the history of mathematics. In the third place, the student is instructed in the importance of mathematics today, and is taught the varying amounts of mathematics which are required for the different professions and occupations. In addition to this program, every effort is made to get the student to have a liking for the subject. In the experience of the writer, several students have finished the course only to change their program and major

* Karnes, H. T., Preparation of Teachers of Secondary Mathematics, *The Mathematics Teacher*, Vol. XXXVIII, p. 3.

** Allendoerfer, C. B., Mathematics for Liberal Arts Students, this MONTHLY, Vol. 54, 1947, p. 573.

Newsom, C. V., A Course in College Mathematics for a Program of General Education, *The Mathematics Teacher*, Vol. XLII, January, 1949.

Ore, Oystein, Mathematics for Students of the Humanities, this MONTHLY, Vol. 51, 1944, p. 453.

in mathematics.

There are some who would criticize strongly such a course as this one. However, there is a rather common belief that the students know as much mathematics, if not more, than those who take three semester hours each of algebra and trigonometry. In addition, they know far more about mathematics and have broadened their cultural background a great deal.

In closing, let us think for a moment what this indoctrination will mean. Miss Smith, teacher of literature, will tell Johnny to learn all the mathematics he can, for one cannot be a creative engineer without it. Miss Jones, teacher of public school music, will tell Tom, who wishes to become an economist, that certain courses of mathematics are most essential if he is to succeed in his chosen field. And so it will continue, ever broadening in scope. Should we pull a pressure act for three units of mathematics? It will not be necessary. Of course, all of this will not take place in two or three years, but ten, fifteen or twenty years will see a great difference in the favor of mathematics.

One last word. This course is beginning to be used by students of the liberal arts for their required six hours. One can find among this group prospective school board members, and most certainly prospective parents, who have not helped the situation in the past by saying in the presence of their children, "I just never could get mathematics, and it certainly does no one any good."

GENERALIZED TERNARY CONTINUED FRACTIONS

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1. Statement of the problem. In [1], J. M. Barbour explained that a Diophantine problem arises in connection with musical scales, namely, to find small integers A, B, and C that stand nearly in the ratio log (5/4), log (3/2), and log 2. Such integers A, B, C are called tuning ratios. Barbour pointed out that this is the same as requiring

$$\frac{\log (5/4)}{\log 2} - \frac{A}{C} \tag{1}$$

$$\frac{\log (3/2)}{\log 2} - \frac{B}{C} \tag{2}$$

to be simultaneously small compared to $1/C$. Accordingly, he attacked the problem by the use of ternary continued fractions, but found it necessary to generalize the ternary continued fraction algorithm quite a bit in order to get a large number of tuning ratios.

Actually, problems of this sort can be solved by use of the generalized Euclidean algorithm. Full instructions for solving just exactly this sort of problem are

set forth in [2], and if these instructions are followed in the present case, practically every tuning ratio given in Barbour's composite table will be found, together with many that are not in Barbour's table.

The purpose of the present note is to give another algorithm for finding tuning ratios. This algorithm has many points in common with the ternary continued fraction algorithm, so much so that it can be considered as a sort of generalization of the ternary continued fraction algorithm. Moreover our algorithm is in a form where extension to the problem of the simultaneous approximation of N irrationals is immediate.

One feature of this algorithm is that the rate of convergence seems to be under the control of the computer. The question of the excellence of the approximations which result from the algorithm is not studied, but the numerical evidence in this particular case suggests that the approximations will generally be very good.

2. The algorithm. Our algorithm is based on the algorithm disclosed in [3]. In [3], let us relax our restrictions on a_n and allow the a_n's to be any set of real numbers of which at least two are incommensurable. Then Theorem 2 holds even after removal of the condition

$$|W_1| + |W_2| + |W_3| + |W_4| > 1,$$

since this condition was used only to insure that at least two of the W's were not zero. This conclusion on the W's now follows from the incommensurability condition, since, if all but one W were zero, we could solve

$$a_1\beta_{i1} + a_2\beta_{i2} + a_3\beta_{i3} + a_4\beta_{i4} = W_i \tag{3}$$

for the a's, and find that the a's are all integer multiples of the non-zero W, thus contradicting our condition that at least two a's must be incommensurable.

When generalized in this fashion, Theorem 2 furnishes the basic step in an algorithm whereby we can find sets of integers β_{ij} with $|\beta_{ij}| = \pm 1$ and successively smaller values of $|W_1| + |W_2| + |W_3| + |W_4|$, where W_i is defined by (3). (Note that $|\beta_{ij}|$ denotes a determinant, whereas $|W_i|$ denotes an absolute value.) This basic step involves two non-zero W's, and replaces the numerically larger by a new smaller W. In case there are more than two non-zero W's, we have our choice of which pair to use in the step. This choice allows us to control the rate of convergence. If we wish rapid convergence and are not concerned with the rate of increase of the sizes of the β_{ij}, we can always choose that pair of W's which produces the least new W. If we wish a slow rate of increase of the sizes of the β_{ij}, we can always choose that pair of W's which provides the new W of "least" β_{ij}. In general, given some desirable attribute of the W's and β_{ij}, we can always choose that pair of W's for which the new W maximizes the attribute in question.

In order to handle Barbour's problem, we seek β_{ij} with $|\beta_{ij}| = \pm 1$ and $|W_i|$ small, where

$$\beta_{i1} \log \tfrac{5}{4} + \beta_{i2} \log \tfrac{3}{2} + \beta_{i3} \log 2 = W_i. \tag{4}$$

Further, we will wish "slow" convergence, in that we are willing for the W_i to go to zero slowly if thereby we keep L_i from increasing rapidly, where

$$L_i = (\beta_{i1})^2 + (\beta_{i2})^2 + (\beta_{i3})^2. \tag{5}$$

We will explain later how to use the β_{ij} to find tuning ratios, but for now let us look at the algorithm for deriving the β_{ij}. In Table I (at the end of the paper), we show the initial steps. We choose the first three W's quite simply. Then our algorithm furnishes the subsequent W's. A record of the definitions of these subsequent W's is appended to the bottom of Table I. The values in the column headed 100 W_i are only approximate, and are computed as follows. For the first three entries, we have simply recorded the integral portion of 100 W_i. Subsequent entries are computed from those above by the definitions at the bottom of the table. Needless to say, accumulation of rounding off errors renders our values of 100 W_i fairly inaccurate as i increases. However, the value of 100 W_i is merely a guide to the operation of the algorithm, and little accuracy is needed.

The important thing is to see the motivation for our definition of the W_i's after the first three. For instance, let us consider the definition of W_5. We have replaced W_3 by W_4, which is $W_3 - W_2$. So we have W_1, W_2, and W_4 to work with. Using pairs of these, we could replace W_2 by any of $W_2 - W_1$, $2W_1 - W_2$, $3W_1 - W_2$, $W_2 - W_4$, or $2W_4 - W_2$, and we could replace W_4 by $W_4 - W_1$ or $2W_1 - W_4$. Of these seven combinations, $W_2 - W_1$ makes L_i the least. So we take W_5 to be $W_2 - W_1$. We then have W_1, W_4, and W_5 to work with. Again, we have the choices of replacing W_4 by $W_4 - W_5$ or $2W_5 - W_4$ or $3W_5 - W_4$ or $W_4 - W_1$ or $2W_1 - W_4$ or replacing W_1 by $W_1 - W_5$ or $2W_5 - W_1$. Of these, $W_4 - W_1$ makes L_i the least, and we choose $W_6 = W_4 - W_1$.

In short, our guiding principle is to choose at each step that W_i derived from two W's which makes L_i a minimum. If two such W_i give the same minimum L_i, we choose that one which gives the smaller value of W_i. This procedure was followed carefully in computing Tables I, II, and III. One could perhaps slow the convergence a bit more by allowing the new W_i to be a linear combination of all three W's. However, it is not clear that this would be worth the considerable amount of extra computation involved in selecting at each step the W_i of minimum L_i.

Because of the accumulating inaccuracies in 100 W_i toward the end of Table I, we start afresh in Table II with W_7, W_9, and W_{10}, these being the W's which are current at the end of Table I. The first three values of 1000 W_i are rounded off from an accurate numerical calculation, but subsequent values are derived from these, so that rounding off errors accumulate again. So after a bit we start in again in Table 3 with three values of $10^4 W_i$ computed afresh.

Let us now see how our β_{ij}'s furnish tuning ratios. For instance consider the first three lines of Table III. They tell us that

$$(6) \qquad -\log \tfrac{5}{4} + 4 \log \tfrac{3}{2} - 2 \log 2 = 0.0124,$$

$$(7) \qquad -2 \log \tfrac{5}{4} - 4 \log \tfrac{3}{2} + 3 \log 2 = 0.0113,$$

$$(8) \qquad 6 \log \tfrac{5}{4} - 5 \log \tfrac{3}{2} + \log 2 = 0.0047.$$

The matrix of the coefficients is

$$B = \begin{Vmatrix} -1 & 4 & -2 \\ -2 & -4 & 3 \\ 6 & -5 & 1 \end{Vmatrix},$$

whose determinant is unity, and whose inverse is

$$B^{-1} = \begin{Vmatrix} 11 & 6 & 4 \\ 20 & 11 & 7 \\ 34 & 19 & 12 \end{Vmatrix}.$$

The constants on the right of (6) and (7) are nearly zero. If they were exactly zero, then log (5/4), log (3/2), and log 2 would have to be in exactly the ratio of the 2-rowed minors of the coefficients of (6) and (7), namely 4, 7, 12. As the constants are nearly zero, log (5/4), log (3/2), and log 2 must stand nearly in the ratio 4, 7, and 12. Hence, 4, 7, and 12 are a set of tuning ratios; in fact they are the familiar set used in the well tempered 12-tone scale.

Since the determinant of B is unity, the elements of B^{-1} are the 2-rowed cofactors of B, and so the columns of B^{-1} are just sets of tuning ratios. Indeed the columns of our B^{-1} are all given in Barbour's composite table of sets of tuning ratios.

The columns of our B^{-1} correspond to the convergents of a ternary continued fraction, so that we have here a generalization of the ternary continued fraction algorithm. The closest equivalent to the coefficients of the ternary continued fraction are our definitions of the successive W's which we have appended at the bottoms of Tables I, II, and III. If the columns of the various B^{-1} are to serve as convergents, this suggests that perhaps we can compute the columns of the various B^{-1} by some sort of recursion procedure. This can be done, but it is a bit complicated, since in general we cannot express a given convergent in terms of the two immediately preceding convergents. The right hand columns of Tables I, II, and III embody our recursion scheme for computing successive convergents. We have listed the convergents in Table IV, at the bottom of which are the recursions used in computing the table, where we let U_i denote the vector of components A_i, B_i, C_i.

To see how we derived our recursions, let us look at the first three lines of Table III. Let W_i denote the vector of components $\beta_{i1}, \beta_{i2}, \beta_{i3}$ as well as the value given by (4). The 15, 13, 12 at the right hand ends of the first three lines of Table III signify that if we take W_{12}, W_{14}, and W_{15} as the three rows of a matrix B, then U_{15}, U_{13}, and U_{12} will be the three columns of B^{-1}. By looking at the exhibited B and B^{-1}, we see that this is indeed the case.

Now we drop W_{14} and add $W_{16}=W_{14}-W_{15}$. What is the corresponding operation on the U's, and what numbers must we now write at the ends of the W_{12}, W_{15}, and W_{16} lines? The corresponding operation on the U's is as follows. We take W_{14} and W_{15}, which are the two W's which we combined to form W_{16}, and look at the ends of their lines. The numbers appearing there are 13 and 12. So we add the two U's with these subscripts, taking $U_{16}=U_{12}+U_{13}$. Next we dropped W_{14} and added W_{16}. So the 13 which was at the end of the W_{14} line is moved down and put at the end of the W_{16} line. Now W_{15} is the other W which went into making up W_{16}, and we replace the 12 as the end of the W_{15} line by a 16, the number of our new U. (The 18 at the end of the W_{15} line gets added later in a similar fashion.) We now have 15, 16, and 13 at the ends of the W_{12}, W_{15}, and W_{16} lines, and the reader may easily verify that if he forms a matrix B with W_{12}, W_{15}, and W_{16} as rows, then B^{-1} will have U_{15}, U_{16}, and U_{13} as columns. Moreover, a careful study of the behavior of the matrix equation $BB^{-1}=1$ will indicate that if we manipulate the rows of B in the manner in which we manipulated the W's to get W_{16}, then we must manipulate the columns of B^{-1} in exactly the manner in which we manipulated the U's, as described above. By the process used, it may be seen that each matrix B is obtained from its predecessor C by subtracting row i from row j and transposing the newly created row to be the bottom row. In the case where $j=3$, no transposition is needed, and we see that $B=PC$, where P is the matrix obtained from the identity matrix I by subtracting row i from row j. Then

$$B^{-1} = (PC)^{-1} = C^{-1}P^{-1}.$$

But P^{-1} is obtained from I by adding row i to row j, and so B^{-1} is obtained from C^{-1} by adding column i to column j. For the cases $j=1$ and $j=2$ an analogous argument can be given.

Since we always form our new W's as differences of two previous W's, we will always form the new U's as sums of two previous U's, and by the procedure indicated above, we can tell at each step which U's go with the current set of three W's. We will go through the next step to illustrate the method once more. The next step is to drop W_{12} and add $W_{17}=W_{12}-W_{16}$. So we define $U_{17}=U_{13}+U_{15}$, since 15 and 13 appear at the ends of the W_{12} and W_{16} lines. The 15 on the end of the W_{12} line is moved down to the end of the W_{17} line when we drop the W_{12} and add the W_{17}. Finally, since W_{16} is the other W involved in this transaction, we put 17, the number of our new U, at the end of the W_{16} line, superseding the 13 which was at the end of that line.

In actual practice, one draws a line through a given W when it is dropped, so that at any time we know which three W's we are dealing with, since they are the only W's not marked out. Moreover, numbers like the 18 at the end of the W_{15} line, which will be added later, have not yet appeared, so that the final numbers then appearing at the ends of our three W lines are the numbers of the corresponding U's. Under these circumstances, our procedure for manipulating the U's is wholly mechanical and can be carried out very quickly. For in-

stance, only three hours were required for the entire computation recorded in Tables I to IV.

A comparison will show that our Table IV compares favorably with Barbour's Table IV, which is a compilation of tuning ratios from many sources.

Quite clearly our algorithm can be used to generate sets of integers standing nearly in the ratios $a_1, a_2, \cdots, a_n$, where the a's are any set of real numbers. If we find $N_0, N_1, \cdots, N_n$ standing nearly in the ratios $1, a_1, a_2, \cdots, a_n$, then we have nearly

$$a_i = \frac{N_i}{N_0},$$

so that our algorithm can be used to approximate several real numbers simultaneously by rationals with the same denominator.

TABLE I

i	β_{i1}	β_{i2}	β_{i3}	$100\ W_i$	
1	1	0	0	22	1, 5, 6
2	0	1	0	40	2, 4
3	0	0	1	69	3
4	0	−1	1	29	3
5	−1	1	0	18	4, 7
6	−1	−1	1	7	3, 8
7	2	−1	0	4	6, 9, 10
8	0	2	−1	11	7
9	−3	0	1	3	8
10	−2	3	−1	7	7

$W_4 = W_3 - W_2$, $W_5 = W_2 - W_1$, $W_6 = W_4 - W_1$, $W_7 = W_1 - W_5$, $W_8 = W_5 - W_6$, $W_9 = W_6 - W_7$, $W_{10} = W_8 - W_7$.

TABLE II

i	β_{i1}	β_{i2}	β_{i3}	$1000\ W_i$	
7	2	−1	0	41	10, 12
9	−3	0	1	24	8, 11, 13
10	−2	3	−1	77	7
11	1	3	−2	53	7
12	−1	4	−2	12	7, 14, 15
13	5	−1	−1	17	12
14	−2	−4	3	12	13
15	6	−5	1	5	12

$W_{11} = W_{10} - W_9$, $W_{12} = W_{11} - W_7$, $W_{13} = W_7 - W_9$, $W_{14} = W_9 - W_{12}$, $W_{15} = W_{13} - W_{12}$.

TABLE III

i	β_{i1}	β_{i2}	β_{i3}	$10^4 W_i$	15
12	− 1	4	− 2	124	
14	− 2	− 4	3	113	13
15	6	− 5	1	47	12, 16, 18
16	− 8	1	2	66	13, 17
17	7	3	− 4	58	15
18	1	8	− 5	11	15, 19, 20, 22
19	− 9	− 7	7	55	17
20	5	−13	6	36	18, 21
21	−14	6	1	19	17
22	−15	− 2	6	8	17, 23, 24, 26, 28
23	16	10	−11	3	22, 25, 27
24	20	−11	0	28	21
25	4	−21	11	25	21
26	19	−19	5	17	21
27	3	−29	16	14	21
28	18	−27	10	6	21

$W_{16}=W_{14}-W_{15}$, $W_{17}=W_{12}-W_{16}$, $W_{18}=W_{17}-W_{15}$, $W_{19}=W_{16}-W_{18}$, $W_{20}=W_{15}-W_{18}$, $W_{21}=W_{19}-W_{20}$, $W_{22}=W_{21}-W_{18}$, $W_{23}=W_{18}-W_{22}$, $W_{24}=W_{20}-W_{22}$, $W_{25}=W_{24}-W_{23}$, $W_{26}=W_{25}-W_{22}$, $W_{27}=W_{26}-W_{23}$, $W_{28}=W_{27}-W_{22}$.

TABLE IV

i	A_i	B_i	C_i	i	A_i	B_i	C_i
1	1	0	0	15	11	20	34
2	0	1	0	16	10	18	31
3	0	0	1	17	17	31	53
4	0	1	1	18	21	38	65
5	1	1	1	19	28	51	87
6	1	1	2	20	49	89	152
7	1	2	3	21	38	69	118
8	1	2	4	22	66	120	205
9	2	3	6	23	83	151	258
10	3	5	9	24	121	220	376
11	2	4	7	25	104	189	323
12	4	7	12	26	159	289	494
13	6	11	19	27	142	258	441
14	7	13	22	28	197	358	612

$U_4=U_2+U_3$, $U_5=U_4+U_1$, $U_6=U_3+U_5$, $U_7=U_4+U_6$, $U_8=U_3+U_7$, $U_9=U_6+U_8$, $U_{10}=U_7+U_9$, $U_{11}=U_7+U_8$, $U_{12}=U_7+U_{10}$, $U_{13}=U_{11}+U_{12}$, $U_{14}=U_7+U_{13}$, $U_{15}=U_{12}+U_{14}$, $U_{16}=U_{12}+U_{13}$, $U_{17}=U_{13}+U_{15}$, $U_{18}=U_{15}+U_{16}$, $U_{19}=U_{16}+U_{17}$, $U_{20}=U_{18}+U_{19}$, $U_{21}=U_{17}+U_{18}$, $U_{22}=U_{17}+U_{20}$, $U_{23}=U_{17}+U_{22}$, $U_{24}=U_{21}+U_{23}$, $U_{25}=U_{21}+U_{22}$, $U_{26}=U_{21}+U_{24}$, $U_{27}=U_{21}+U_{25}$, $U_{28}=U_{21}+U_{26}$.

References

1. J. M. Barbour, Music and ternary continued fractions, this MONTHLY, vol. 55, 1948, pp. 545–555.

2. Barkley Rosser, A generalization of the Euclidean algorithm to several dimensions, Duke Math. Jour., vol. 9, 1942, pp. 59–95.

3. Barkley Rosser, A note on the linear diophantine equation, this MONTHLY, vol. 48, 1941, pp. 662–666.

Editorial Note on the Preceding Paper. After the preceding paper had been accepted, Viggo Brun of the University of Oslo called attention to the fact that he published a paper in 1919 that gives the same numbers as those found by Barbour; the specific reference to Brun's paper is as follows: En generalisation av kjedebrøken (avec un résumé en français) Videnskapsselskapets skrifter, Oslo 1919 og 1920. The method of Brun appears to be a special case of that presented by Rosser.

A DOUBLE SERIES SUMMED GEOMETRICALLY

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L. R. Ford* has used a certain diagram of circles to prove geometrically some results in the theory of numbers. We shall derive another diagram from his and use it to prove the following:

THEOREM. *If $f(p)$ is a monotone function of the positive integers and* $\lim_{p\to\infty} f(p) = \lambda$, *then*

$$(1) \qquad \sum_{\substack{p,q=1 \\ (p,q)=1}}^{\infty} \left[\frac{f(q)}{p(p+q)} + \frac{f(p)}{q(p+q)} - \frac{f(p+q)}{pq} \right] = 2(f(1) - \lambda).$$

Ford's diagram consists of circles with centers at $(u/p,\ 1/2p^2)$ and radius $1/2p^2$, where u/p ranges over all rational numbers expressed in lowest terms, *i.e.*, u and p are integral, $p>0$ and $(u, p)=1$. These circles are non-overlapping and are tangent to the x-axis. The circles corresponding to u/p and v/q are tangent if and only if $uq-vp=\pm 1$ and the circle tangent to both of these and having its center below the line of centers corresponds to $(u+v)/(p+q)$.

Restrict u/p to lie on the closed interval $0 \leqq u/p \leqq 1$. Join the centers of tangent circles with line segments and as a result the rectangular area between the lines $y=0$ and $y=1/2$ and between the lines $x=0$ and $x=1$ is completely filled with non-overlapping triangles.

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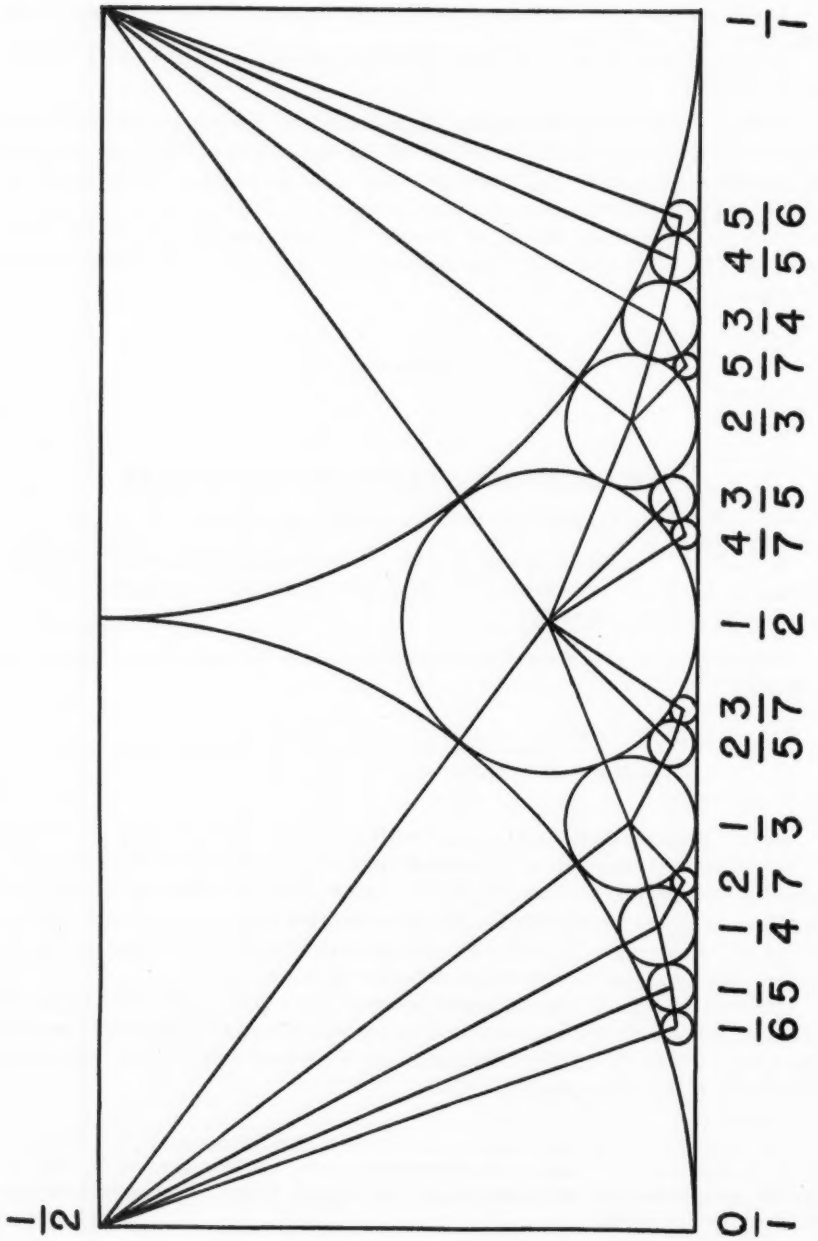
* L. R. Ford, Fractions, this MONTHLY, vol. 45, 1938, pp. 586–601. See also G. T. Williams and D. H. Browne, A family of integers and a theorem on circles, this MONTHLY, vol. 54, 1947, pp. 534–536. For a bibliography on the diagram of circles see J. F. Koksma, Diophantische Approximationen, 1936, pp. 42–43.


1/1
5/6
4/5
3/4
5/7
2/3
4/7
3/5
1/2
5/7
2/3
3/7
1/2
4/7
1/2
5/6
1/1
0/1
-1/2

If instead of the centers $(a/b, 1/2b^2)$ we use as vertices $(a/b, f(b))$ the new triangles are likewise non-overlapping and entirely fill up the area between $f(1)$ and $\lambda = \lim_{b\to\infty} f(b)$. All that has occurred is a distortion of the scale on the y-axis including possibly a reflection.

We now compute the area of a single triangle. If u/p, v/q, and $(u+v)/(p+q)$ are the abscissas of the vertices of a triangle, the area of the triangle is, except for sign,

$$\frac{1}{2}\begin{vmatrix} u/p & f(p) & 1 \\ v/q & f(q) & 1 \\ (u+v)/(p+q) & f(p+q) & 1 \end{vmatrix} = \pm\frac{1}{2}A(p, q; f),$$

where $A(p, q; f)$ is the expression within brackets in (1). If $f(p)$ is decreasing then $A \geqq 0$, for, if $f(p)$ and $f(q)$ are decreased to $f(p+q)$ in A, the expression becomes zero. Similarly if $f(p)$ is increasing, $A \leqq 0$. Because the terms A in the series (1) all have the same sign, all of the areas are being added positively or all negatively. Also the value of the series is independent of the order of summation.

We prove finally that the summation (1) accounts for precisely all the triangles.

By interchanging u/p and v/q the sign of $uq-vp$ changes. Thus we need to show only that for every pair p, q with $(p, q)=1$ there is one and only one solution of

$$uq - vp = 1 \tag{2}$$

with

$$0 \leqq u \leqq p, \qquad 0 \leqq v \leqq q, \qquad 0 < p, \qquad 0 < q. \tag{3}$$

The existence of a solution is well-known. To show the uniqueness let, u', v' be another solution. Then $u'q-v'p=1=uq-vp$, so that $(u-u')q=(v-v')p$. If $u=u'$ then $v=v'$ and the solutions are the same. But if $u\neq u'$, then since $(p, q)=1$, p divides $u-u'$ and by virtue of (3) this is possible only when $u=0$ and $u'=p$ (or $u=p$ and $u'=0$). But if $u=0$, by (2), $-vp=1$; this is impossible since $vp \geqq 0$. The proof of the theorem is now complete.

If in (1) we did not require $(p, q)=1$, then the result is

$$\sum_{p,q=1}^{\infty} A(p, q; f) = 2\sum_{r=1}^{\infty} [f(r) - \lambda]/r^2. \tag{4}$$

This follows from the fact that summing over all positive integral pairs p, q is equivalent to summing over all pr, qr where $(p, q)=1$ and $r=1, 2, \cdots$. Now for fixed r,

$$\sum_{(p,q)=1}\left[\frac{f(qr)}{pr(pr+qr)}+\frac{f(pr)}{qr(pr+qr)}-\frac{f(pr+qr)}{prqr}\right]$$

$$=\frac{1}{r^2}\sum_{(p,q)=1}A(p,q;g)=2[g(1)-\mu]/r^2,$$

where $g(p)=f(pr)$ and $\mu=\lim_{p\to\infty}g(p)$. Hence $g(1)=f(r)$ and $\lambda=\mu$, and (4) follows immediately. Both the theorem and (4) may be proved by manipulation of the series directly.

Every function $h(p)$ defined for positive integral p is a sum of two monotone functions h_i and h_d where

$$h_i(p)=\tfrac{1}{2}h(1)+\sum_{s=2}^{p}\max(0,\ h(s)-h(s-1)),$$

$$h_d(p)=\tfrac{1}{2}h(1)+\sum_{s=2}^{p}\min(0,\ h(s)-h(s-1)).$$

If either h_i or h_d is bounded then in the theorem and in equation (4), we may replace f by h, thus obtaining a more general result.

In the proof of (4) it was possible to ignore the order of summation because the terms all have the same sign. One may not, however, separately sum each of the terms of $A(p, q; f)$; *e.g.*, if $f(p)=1$ the series

$$\sum_{(p,q)=1}f(p)/q(p+q)\quad\text{is divergent whereas,}\quad\sum A(p,q;f)=0.$$

A few special cases of the theorem may be of interest:

$$\sum_{(p,q)=1}\frac{1}{pq(p+q)}=2,\qquad\text{by taking } f(p)=1/p,$$

$$\sum_{(p,q)=1}\frac{p^2+pq+q^2}{p^2q^2(p+q)^2}=2,\qquad\text{by taking } f(p)=1/p^2,$$

and using (4), we get corresponding results

$$\sum_{p,q}\frac{1}{pq(p+q)}=2\zeta(3),\qquad\sum_{p,q}\frac{p^2+pq+q^2}{p^2q^2(p+q)^2}=2\zeta(4)=\frac{\pi^4}{45}$$

where $\zeta(n)=1/1^n+1/2^n+\cdots$.

By taking $f(p)=1/p^m$ and by means of other considerations, it is possible to evaluate the double harmonic series

$$\sum_{p,q=1}^{\infty}p^{-r}q^{-s}(p+q)^{-t}$$

in terms of the harmonic series $\zeta(n)$ in many cases; *e.g.*, if $r+s+t$ is odd. However $\zeta(n)$ is known only when n is even.

MATHEMATICAL NOTES

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DISCONTINUOUS FUNCTIONS WITH THE DARBOUX PROPERTY

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A real-valued $f(x)$ has the Darboux property if: whenever $f(x_1)<u<f(x_2)$ then $f(x)=u$ for at least one x between x_1 and x_2. Such $f(x)$ need not be continuous[1] nor even Riemann-integrable[2] and may, in fact, be discontinuous for all x.[3] We shall construct such an $f(x)$ which actually takes on every real number as value a continuum number of times on every interval; it is non-Lebesgue measurable on every measurable set of positive measure and hence does not coincide on any such set with any continuous function.

Let $a_1, a_2, \cdots, a_\alpha, \cdots (\alpha<\Omega)$ be a Hamel basis, so that every real number has a unique expression $x=\sum_\alpha r_\alpha a_\alpha$, with rational r_α, of which only a finite number differ from zero.[4] Rearrange this basis into a two-fold transfinite sequence $b_1, b_2, \cdots, b_\alpha, \cdots (\alpha<\Omega)$; $c_1, c_2, \cdots, c_\beta, \cdots (\beta<\Omega)$, and for $x=\sum_\alpha r_\alpha b_\alpha + \sum_\beta s_\beta c_\beta$ set $f(x)=\sum_\alpha r_\alpha a_\alpha$.

For any interval (a, b) and any $u=\sum_\alpha r_\alpha a_\alpha$ we will have $f(x)=u$ if $x=\sum_\alpha r_\alpha b_\alpha+\sum_\beta s_\beta c_\beta$ for all choices of the s_β; in particular, if only one $s_\beta\neq 0$, say s_{β_0}, and it is chosen so that $a-\sum_\alpha r_\alpha b_\alpha<s_{\beta_0}c_{\beta_0}<b-\sum_\alpha r_\alpha b_\alpha$. Hence $f(x)=u$ for continuum many x in (a, b). Since this $f(x)$ is a discontinuous solution of the functional equation $g(x+y)=g(x)+g(y)$, it is not Lebesgue measurable on any set of positive measure.[5]

Using another construction we can define an $f(x)$ with the stronger property that it takes on every real number as value a continuum number of times on every perfect set, including those of measure zero. Let Ω_1 be the smallest ordinal number with continuum many predecessors. The real numbers can be arranged as a transfinite sequence $u_1, u_2, \cdots, u_\alpha, \cdots (\alpha<\Omega_1)$. There are continuum many perfect sets and they can be arranged as a transfinite sequence $P_1, P_2, \cdots, P_\alpha, \cdots (\alpha<\Omega_1)$, so that every perfect set appears continuum many times. Now for each α define p_α, by induction on α, to be the first u not previously selected which is contained in P_α; such a u will exist since, for each α, less than continuum many u's will have been selected previously, and the perfect set P_α contains continuum many numbers. This process will select continuum many p's from each perfect set P since each P appears continuum many

[1] G. Darboux, Annales Scientifiques de l'Ecole Normale Supérieure, 2nd series, vol. 4 (1875), pp. 57–112.

[2] V. Volterra, Giornale de Battaglini, 1881.

[3] H. Lebesgue, Leçons sur l'intégration, Gauthier-Villars, Paris, 1904, page 90. Lebesgue's function is zero except on a set of measure zero.

[4] G. Hamel, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[5] A. Ostrowski, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 38, 1929.

times as P_α and of course the p's selected from different perfect sets will be mutually exclusive.[6] Now for each perfect set let its selected p's be arranged as a double transfinite sequence $x_{\alpha\beta}(\alpha, \beta<\Omega_1)$, and set $f(x_{\alpha\beta})=u_\alpha$. This $f(x)$ clearly takes on every real number as value a continuum number of times on every perfect set. If $f(x)$ were Lebesgue measurable on any set of positive measure there would be a finite K with $f(x)<K$ on a set of positive measure and hence on a perfect subset (of positive measure). This contradiction shows that $f(x)$ cannot be measurable on any set of positive measure.

DECOMPOSITION OF AN INFINITE-DIMENSIONAL LINEAR SYSTEM INTO UBIQUITOUS CONVEX SETS

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1. **Introduction.** Suppose that L is a (real) linear system (that is, a module over the real number field). A maximal linearly independent subset of L is called a *Hamel basis* for L. It is well known [2] that L must have a Hamel basis and that all its Hamel bases have the same cardinal number $d(L)$, which is called the *dimension* of L.

If x and y are distinct points of L, $[x, y]$ will denote the line segment including its endpoints, $(x, y]\equiv[x, y]-\{x\}$, and so on. For a convex subset X of L, lin X will denote the union of X and the set of all points $y\in L$ such that $(y, x]\subset X$ for some x; X will be called *ubiquitous* if lin $X=L$ (that is, if each point of L is linearly accessible from X). If L is n-dimensional, it is isomorphic with Euclidean n-space and (with X convex) lin X is merely the closure of X, so L has no ubiquitous convex proper subsets. For infinite-dimensional systems, on the other hand, the situation is quite different.

2. **Theorem.** We shall establish the following result.

THEOREM. *Suppose that L is an infinite-dimensional linear system and N is a cardinal number such that $2\leqq N\leqq d(L)$. Then L can be expressed as the union of N pairwise disjoint ubiquitous convex sets.*

Proof. Let H be a Hamel basis for L. Since $d(L)\geqq\aleph_0$, H can be expressed as the union of $N-1$ pairwise disjoint infinite sets; say $H=\bigcup_{1\leqq i<M} H_i$, where M is an ordinal number having $N-1$ predecessors $1, 2, \cdots$. Each set H_i can be assigned an anti-reflexive linear ordering $<_i$ with respect to which it has no last element.

Now each point $x\in L$ can be expressed in the form $\sum_{h\in H} a(h, x)h$, where all but finitely many of the coefficients $a(h, x)$ are zero. For each i and x, define $h_i(x)$ as follows: if $a(h, x)$ is non-zero for some $h\in H_i$, then $h_i(x)$ is the last such h (in terms of the ordering $<_i$); otherwise (that is, if $h\in H_i$ implies $a(h, x)=0$), $h_i(x)$ is an arbitrary but fixed point of H_i. For each i and x, let $b_i(x)=a(h_i(x), x)$.

[6] This construction was used by W. Sierpinski and N. Lusin, Comptes Rendus (Paris), vol. 165 (1917), pp. 422–424, to subdivide an interval into continuum many parts each of exterior Lebesgue measure equal to the length of the interval.

Let $C_0=\{x\,|\,b_i(x)\leqq 0$ for all i's$\}$ and for $1\leqq n<M$ define $C_n=\{x\,|\,b_i(x)\leqq 0$ whenever $i<n$, but $b_n(x)>0\}$. It can be verified that the sets $C_i(0\leqq i<M)$ are pairwise disjoint, cover L, and are all convex. (In fact, each C_n is a convex cone with vertex at the origin, although only C_0 contains the origin.) It remains to show that each set C_n is ubiquitous.

To this end, consider an arbitrary point $x\in L$, an arbitrary index n, and let $K'=\{i\,|\,b_i(x)\neq 0\}$. Let $K=K'\cup\{n\}$ if $n\neq 0$ and $K=K'$ if $n=0$. For each $i\in K$ pick $y_i\in H_i$ such that $h_i(x)<_i y_i$. Let $\epsilon_n=1$, $\epsilon_i=-1$ for $i\neq n$, and define $y=\sum_{i\in K}\epsilon_i y_i$. (Note that K is a finite set.) It can be verified that $x+ty\in C_n$ for each $t>0$, so $(x,\ c+y]\subset C_n$, $x\in\text{lin } C_n$, and the proof is complete.

Notice that if any topology be assigned to L in which every line is a continuous image (under the natural mapping) of the real line, then each ubiquitous convex set is dense. For certain linear spaces and with "ubiquitous" replaced by "dense," the above theorem was proved in [1]. However, the sets obtained in [1] were not ubiquitous and the method of [1] required the existence of a dense *linear* subset, which is not needed here. Hence the present theorem is applicable (to obtain dense convex subsets) to many spaces not covered in [1].

As is pointed out in [1], for a large class of spaces L, $d(L)$ is equal to the cardinal number of L.

The author first became acquainted with ubiquitous convex sets through the comments of a referee of one of his papers.

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ASYMPTOTIC BEHAVIOR OF MEAN VALUE FUNCTIONS

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1. Introduction. The mean value function $M(t)$ of order t of n positive real numbers $x_\nu(\nu=1,\ \cdots,\ n)$, weighted by n positive real numbers $a_\nu(\nu=1,\ \cdots,\ n)$ with $\sum a_\nu=1$, is defined for all real arguments t except $t=-\infty,\ 0,\ +\infty$ by the formula

$$M(t)=\left(\sum a_\nu x_\nu^t\right)^{t^{-1}}.$$

We mention briefly some properties of this well-known function; for a detailed discussion see [1, 2]. At its three singular points, $M(t)$ can be defined so as to become continuous. It is clearly analytic elsewhere. Furthermore, in justification of the term mean value function, $M(t)$ satisfies the inequality

$$\min x_\nu\leqq M(t)\leqq\max x_\nu.$$

Thus $M(t)=x$ for all t if $x_\nu=x$ for all ν. If, however, not all the x_ν are equal, then $M(t)$ steadily increases from the value $\min x_\nu$ at $t=-\infty$ to the value $\max x_\nu$ at $t=+\infty$, with $M'(t)>0$ for $-\infty<t<+\infty$ [2].

In this note we investigate the exact behavior of $M(t)$ at its three singular points.

2. Behavior at zero. The mean value function $M(t)$ may be expressed in the form

$$M(t) = S(t)^{t^{-1}},$$

where

$$S(t) = \sum a_\nu x_\nu^t.$$

Clearly $S(t)$ is analytic at $t=0$, and

$$S(0) = \sum a_\nu = 1;$$

so $\log S(t)$ is also analytic at $t=0$ and $\log S(0)=0$. Therefore

$$\log M(t) = t^{-1} \log S(t)$$

and, consequently, $M(t)$ is analytic at $t=0$ with removable singularities there, these singularities being removed by defining

$$\log M(0) = (\log S)'(0) = S'(0) = \sum a_\nu \log x_\nu.$$

3. Asymptotic equivalence. Let ϕ be a formal power series in an indeterminate σ with real coefficients:

$$\phi = \sum_{h=0}^{\infty} c_h \sigma^h.$$

We say that a function f defined in a neighborhood of $t_0 (-\infty \leqq t_0 \leqq +\infty)$ is weakly asymptotic to ϕ as $t \to t_0$, and write $f \sim \phi$ as $t \to t_0$, if

$$f(t) = \sum_{h=0}^{k} c_h s^h + o(s^k) \qquad (t \to t_0,\ s \to 0),$$

for $k=0, 1, 2, \cdots$, where

$$s = \begin{cases} t - t_0 & \text{if} \quad t_0 \neq \omega, \\ t^{-1} & \text{if} \quad t_0 = \omega, \end{cases}$$

and ω stands for $-\infty$ or $+\infty$. Furthermore, we say that f is strongly asymptotic to ϕ as $t \to t_0$, and write $f \approx \phi$ as $t \to t_0$, if, for $k=0, 1, 2, \cdots$, the kth derivative $f^{(k)}$ of f is weakly asymptotic as $t \to t_0$ to the kth formal derivative $\phi^{(k)}$ of ϕ.

The relations of asymptotic equivalence as defined above subsist between a function and a formal power series. These notions may be extended as follows. We call two functions f and g weakly or strongly asymptotically equivalent as $t \to t_0$, and write $f \sim g$ or $f \approx g$ as $t \to t_0$, provided that f and g are both weakly or are both strongly asymptotic in the original sense as $t \to t_0$ to the same formal power series.

It can be shown that both weak and strong asymptotic equivalence are preserved under addition, multiplication, function composition, and integration, and that strong asymptotic equivalence is preserved under differentiation.

In applications we shall extensively employ these preservation properties of strong asymptotic equivalence. But we cannot operate in a void: one strong asymptotic expansion formula is required, namely: $\theta^t \approx 0$ as $t \to \infty$ if $0 < \theta < 1$, this equivalence being a consequence of the limit relations $t^k\theta^t \to 0$ as $t \to \infty$ for $k = 0, 1, 2, \cdots$.

4. Behavior at infinity. We are here concerned with the strong asymptotic behavior of $M(t)$ as $t \to \omega$. Since $M(t) = x$ for all t if $x_\nu = x$ for all ν, we may assume that $n \geqq 2$ and that $x_1 < \cdots < x_n$. For notational convenience we define a_ω and x_ω as follows:

$$a_{-\infty} = a_1 \qquad a_{+\infty} = a_n$$
$$x_{-\infty} = x_1 \qquad x_{+\infty} = x_n,$$

and let ω be attached as a subscript to inequality signs with the meaning that the inequality is to be reversed if $\omega = -\infty$, but is to be preserved if $\omega = +\infty$. It is understood without explicit mention that $t \to \omega$ in limits and asymptotic expansions.

Consider the following functions of t:

$$M_\omega(t) = x_\omega a_\omega^{t^{-1}},$$
$$I_\omega(t) = S_\omega(t)^{t^{-1}},$$
$$S_\omega(t) = \sum \left(\frac{a_\nu}{a_\omega}\right)\left(\frac{x_\nu}{x_\omega}\right)^t,$$

and note that

$$M(t) = M_\omega(t) I_\omega(t).$$

As we have mentioned, $\theta^t \approx 0$ for $0 < \theta <_\omega 1$, so

$$S_\omega(t) = 1 + \text{terms of type } \theta^t \approx 1,$$
$$I_\omega(t) = \exp\,(t^{-1} \log S_\omega(t)) \approx 1.$$

Consequently the desired strong asymptotic expansion of the mean value function $M(t)$ as $t \to \omega$ is

$$M(t) \approx M_\omega(t),$$

the function $M_\omega(t)$ being analytic at $t = \omega$ with $M_\omega(t) \to x_\omega$. It follows that $M(t) \to x_\omega$. Furthermore, $M(t)$ in its approach to x_ω is squeezed between $M_\omega(t)$ and x_ω; for $S_\omega(t) > 1$, hence $I_\omega(t) >_\omega 1$ provided $t >_\omega 0$, so

$$M(t) >_\omega M_\omega(t) \qquad (t >_\omega 0).$$

The logarithm of the expansion for $M(t)$ is

$$\log M(t) \approx \log M_\omega(t) = \log x_\omega + t^{-1} \log a_\omega.$$

Differentiating this $k \geqq 1$ times with respect to t, we get

$$(\log M)^{(k)}(t) \approx t^{-k-1} k!(-1)^k \log a_\omega,$$

whence

$$M^{(k)}(t)/M(t) \approx t^{-k-1} k!(-1)^k \log a_\omega + \cdots .$$

Thus the kth derivatives of both $M(t)$ and $\log M(t)$ are >0 near $t=-\infty$, and near $t=+\infty$ are >0 if k is odd and <0 if k is even. In particular, taking $k=2$, we see that $M(t)$ and $\log M(t)$ *as functions of* t *are convex near* $t=-\infty$ *and concave near* $t=+\infty$. Three of the four parts of this result were recently obtained in quite a different fashion by Shniad [3]; his methods did not yield the concavity of $M(t)$ near $t=+\infty$.

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ON A PAIR OF FUNCTIONAL EQUATIONS

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1. Introduction. It is well known that all non-trivial, real, continuous solutions, defined for all $x>0$, of Cauchy's functional equation

$$F(xy) = F(x)F(y) \tag{1}$$

are of the form $F(x)=x^c$, where c is any real number.[1] In a recent issue of this MONTHLY there was considered the functional equation

$$f(xy) = [f(x)]^{(y^\beta)}[f(y)]^{(x^\beta)}, \tag{2}$$

which reduces to the equation (1) if $\beta=0$.[2] Solutions of equation (2) were found to be of the form $x^{(cx^\beta)}$. In deriving these solutions the differentiability of the solution was assumed. This is an unnecessary assumption, for it follows from Cauchy's result for equation (1) that all real, continuous solutions of (2) which are defined for all $x>0$ are of the form $x^{(cx^\beta)}$, where c is any real constant. To derive this result one needs only to raise both sides of the equation (2) to the power $(xy)^{-\beta}$ and thus obtain

$$f(xy)^{1/(xy)^\beta} = f(x)^{1/x^\beta} f(y)^{1/y^\beta}. \tag{3}$$

[1] A non-trivial solution is one which does not vanish identically. A solution in this paper shall mean a non-trivial solution.

[2] Vol. 56 (1949), p. 414.

This equation is of the form of equation (1) if one takes

$$F(x) = f(x)^{1/x^\beta}.$$

Since all real continuous solutions of (1) are given by x^c, it follows that all real continuous solutions of (2), which are defined for all $x>0$, are of the form $f(x)=x^{(cx^\beta)}$.

It is the object of this note to solve a pair of simultaneous functional equations of which equations (1) and (2) are very special cases.

2. A pair of generalized Cauchy functional equations. We consider the set of functional equations

$$(A) \qquad f(xy) = g(x)^{p(y)}h(y)^{q(x)},$$

$$(B) \qquad r(xy) = p(x)q(y)$$

and look for all real, continuous functions defined for all $x>0$ and satisfying these equations. We must determine all sets of six real, continuous, not identically vanishing functions for which (A) and (B) are satisfied.

We consider first equation (B). In this equation let us set $y=1$. Then $r(x)=p(x)q(1)$. If $q(1)=0$, then $r(x)\equiv 0$. Hence $q(1)\neq 0$, and $p(x)=r(x)/q(1)$. Next let $x=1$; then $r(y)=p(1)q(y)$. Again if $p(1)=0$, then $r(y)\equiv 0$. Hence $q(y)=r(y)/p(1)$. Thus (B) can be rewritten in the form

$$r(xy) = \frac{r(x)r(y)}{p(1)q(1)}.$$

Dividing both sides of this equation by $p(1)q(1)$, we get

$$\frac{r(xy)}{p(1)q(1)} = \frac{r(x)}{p(1)q(1)} \cdot \frac{r(y)}{p(1)q(1)}.$$

In this equation we set $r(x)/[p(1)q(1)]$ equal to $s(x)$. Then $s(x)$ satisfies equation (1); hence $s(x)=x^\beta$ where β is any real number. Hence $p(x)=p(1)x^\beta$ $q(x)=q(1)x^\beta$, $r(x)=p(1)q(1)x^\beta$, and thus all real, continuous solutions of (B) are given by $p(x)=c_1x^\beta$, $q(x)=c_2x^\beta$, $r(x)=c_1c_2x^\beta$, where the c_i $(i=1, 2)$ are any real constants different from zero.

We now turn to equation (A). It can now be written as

$$f(xy) = g(x)^{(c_1y^\beta)}h(y)^{(c_2x^\beta)}.$$

In order to have only real solutions, $f(x)$, $g(x)$, $h(x)$ must be assumed to be positive. Let us write $g(x)^{c_1}=u(x)$, $h(y)^{c_2}=v(y)$. Then the last displayed functional equation takes the form

$$f(xy) = u(x)^{(y^\beta)}v(y)^{(x^\beta)}.$$

Raising both sides of this equation to the power $(xy)^{-\beta}$, we get

$$(A') \qquad f(xy)^{1/(xy)^\beta} = u(x)^{1/(x)^\beta}v(y)^{1/(y)^\beta}.$$

If in this equation we write for the left side $F(xy)$, and let $G(x)$ and $H(y)$ be the first and second factors of the right hand side respectively, then (A') becomes

$$(A'') \qquad F(xy) = G(x)H(y).$$

This, however, is equation (B) which was solved above. Hence all continuous solutions of (A'') are given by $G(x) = k_1x^\alpha$, $H(x) = k_2x^\alpha$, $F(x) = k_1k_2x^\alpha$, where α is any real constant and $k_i > 0$ $(i = 1, 2)$. Substituting these values in the equations which define $f(x)$, $g(x)$, and $h(x)$, we get the result stated in the following theorem.

THEOREM. *All real, continuous solutions, defined for all $x > 0$, of the set of functional equations (A) and (B) are of the form*

$$p(x) = c_1x^\beta, \qquad q(x) = c_2x^\beta, \qquad r(x) = c_1c_2x^\beta,$$
$$g(x) = (k_1x^\alpha)^{x^\beta/c_1}, \qquad h(x) = (k_2x^\alpha)^{x^\beta/c_2},$$
$$f(x) = (k_1k_2x^\alpha)^{x^\beta},$$

where α, β are any real constants while c_i and k_i are any real numbers such that $c_i \neq 0$, $k_i > 0$ $(i = 1, 2)$.

3. Special cases. Cauchy's equation (1) is equivalent to the equations (A) and (B) with the additional simultaneous equations $f(x) = g(x) = h(x)$, and $p(x) = q(x) = r(x) = 1$. The solution of Cauchy's equation is described by a special case of this theorem; namely that for which $\beta = 0$, $c_1 = c_2 = k_1 = k_2 = 1$, α arbitrary. The equation (2) is equivalent to the equations (A) and (B) with the additional simultaneous equations $f(x) = g(x) = h(x)$, and $p(x) = q(x) = r(x) = x^\beta$. The solution of equation (2) is described by the special case of our theorem for which $c_1 = c_2 = k_1 = k_2 = 1$, α and β arbitrary.

COROLLARY. *All real continuous solutions, defined for all $x > 0$, of the set of functional equations,*

$$(C) \qquad F(xy) = p(y)G(x) + g(x)H(y),$$

$$(D) \qquad r(xy) = p(x)g(y),$$

are of the form

$$p(x) = c_1x^\beta, \qquad q(x) = c_2x^\beta, \qquad r(x) = c_1c_2x^\beta,$$
$$G(x) = \frac{x^\beta}{c_1}(\alpha \log x + \log k_1), \qquad H(x) = \frac{x^\beta}{c_2}(\alpha \log x + \log k_2),$$
$$F(x) = x^\beta(\alpha \log x + \log k_1k_2),$$

where α, β are any real numbers while c_i and k_i are any real numbers such that $c_i \neq 0$, $k_i > 0$, $(i = 1, 2)$.

To derive this result from the Theorem we write (C) as

$$e^{F(xy)} = e^{G(x)p(y)}e^{H(y)q(x)},$$

which is equation (A) with $f(x) = e^{F(xy)}$, $g(x) = e^{G(x)}$, and $h(x) = e^{H(x)}$. Taking the logarithms of the solutions for $f(x)$, $g(x)$, and $h(x)$ given in the Theorem we have the result of the Corollary.

By taking logarithms or exponential functions of the functions occurring in the given functional equations one can derive other results similar to those given here.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College,

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NOTE ON AN EXTENSION OF THE METHOD OF UNDETERMINED COEFFICIENTS IN SOLVING A LINEAR DIFFERENTIAL EQUATION

H. E. STELSON, University of Hawaii and Michigan State College

The method of undetermined coefficients for solving a linear differential equation with constant coefficients, as presented in many texts* has the requirement that the method is applicable only if the differential equation

$$F(D)y = (r_0D^s + r_1D^{s-1} + \cdots + r_{s-1}D + r_s)y = f(x) \tag{1}$$

has a right member $f(x)$ such that $f(x)$ and a finite number of its successive derivatives form a set of linear dependent functions. The restriction requires that $f(x)$ satisfy a linear homogeneous differential equation with constant coefficients

$$(b_0D^t + b_1D^{t-1} + \cdots + b_{t-1}D + b_t)f(x) = 0.$$

By the theory for the solution of equations of this type $f(x)$ is a linear combination with constant coefficients of terms of the form

$$x^pe^{qx}, \qquad x^pe^{\alpha x}\cos\beta x, \qquad x^pe^{\alpha x}\sin\beta x \qquad (p \geqq 0) \tag{2}$$

where p, q, α and β are real. Now (2) are essentially functions of the form,

$$Cx^pe^{(\alpha+i\beta)x} \qquad (p \geqq 0).$$

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* Reddick, H. W., Differential Equations, Section 36, 1943. Cohen, Abraham, Differential Equations, Section 50, 1906.

It is the purpose of this paper to replace this restriction by the more general requirement that $f(x)$ be of the form,

$$Cx^p e^{(\alpha+i\beta)x^m} \qquad (p \geqq 0) \ (m \text{ a positive integer}). \tag{3}$$

We present a method for solving equations of this type, when the primitive is expressible in finite form.

Denote the general solution† by

$$y = A_0 f(x) + A_1 f'(x) + A_2 f''(x) + \cdots \qquad (A\text{'s are undetermined constants})$$

or

$$y = e^{(\alpha+i\beta)x^m}(a_0 + a_1 x + a_2 x^2 \cdots) \tag{4}$$

with undetermined constant coefficients, $a_0, a_1, a_2, \cdots$. Substituting (4) in (1) gives an equation,

$$H_0 + H_1 x + H_2 x^2 + \cdots = 0 \tag{5}$$

where H_i involves $a_0, a_1, a_2 \cdots$ and the coefficients of $f(x)$ of form (3). Since the sum of the coefficients of each power of x (i.e. H_i) must equal zero, we have the set of linear equations:

$$H_i = 0 \qquad (i = 0, 1, 2, \cdots) \tag{6}$$

to solve for $a_0, a_1, a_2, \cdots$.

Now if the highest power of x in $f(x)$ of form (3) is p then the equations,

$$H_i = 0 \qquad (i = p+1, p+2, \cdots) \tag{7}$$

will be homogeneous. The linear recurrence relation, derived from substituting (4) in (1) by which equations (7) are derived has at most

$$\gamma \equiv \sum_{i=1}^{m_1} (s + 2 - i) \qquad \text{terms if } m \leqq s+1 \tag{8}$$

or

$$\gamma \equiv \sum_{i=1}^{s} (s+2-i) \qquad \text{terms if } m > s+1.$$

In relation (4), expand $e^{(\alpha+i\beta)x^m}$ in a Maclaurin's series and multiply this expansion by $(a_0+a_1x+a_2x^2 \cdots)$, then if this series is substituted in (1), it can be observed that H_0 has at most the terms $a_0, a_1, \cdots, a_s$; H_1 has at most the terms $a_0, a_1 \cdots a_{s+1}$; and H_i has at most the terms $a_0, a_1, \cdots a_{s+i}$. We now solve the set of linear equations:

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† Coble, A. B., this MONTHLY, vol. 26, p. 12, 1919. Conkwright, N. B., this MONTHLY, vol. 41, p. 228, 1934.

$$H_i = 0 \qquad (i = 0, 1, \cdots, p + \gamma) \tag{9}$$

for $a_0, a_1, \cdots a_{p+\gamma}$.

Since y as given by (4) is equivalent to a power series, by the theory for a power series solution‡ for a differential equation of order s, there are s arbitrary constants in the solution. Hence we can assign s of the a's values for a particular solution. The solution is easier if we assign $a_{p+1}, a_{p+2}, \cdots a_{p+s}$ zero and first solve the first $p+1$ equations** of (9) ($p+1$ non-homogeneous linear equations in $p+1$ unknowns) and use these results to solve the last γ homogeneous equations of (9).

Now if $a_{p+s+1}, a_{p+s+2} \cdots a_{p+\gamma-1}$ are equal to zero in the solution of (9), all but one of the terms in the recurrence relation are zero. Therefore the remaining term is zero. Hence for equations (7) (these equations do not contain $a_0, a_1, \cdots a_p$) $a_{p+1}, a_{p+2}, \cdots$ are all zero. In this case we have a particular solution in finite form. Otherwise the solution will be infinite.

In particular, when $m=1$, (8) has the value $s+1$, and $\gamma-s$ has the value 1, so that the equations (7) always have zero solutions. In this case it is always possible to find a finite solution of form (4). The coefficient of a_0 in H_0 is $F(\alpha+i\beta)$. The determinant of the unknowns is $[F(\alpha+i\beta)]^{p+1}$. If $F(\alpha+i\beta)=0$, and $\alpha+i\beta$ is a root of multiplicity m, we cut out $a_0, a_1, \cdots a_{m-1}$ and proceed as before.

An illustration may be presented. Solve the differential equation

$$y'' - 12y = 2e^{x^2}(4x^4 - 3).$$

The recurrence relation is

$$R_n \equiv (n + 2)(n + 1)a_{n+2} + (4n - 10)a_n + 4a_{n-2}.$$

Hence

$$H_0 = 2a_2 - 10a_0 + 6 = 0$$

$$H_1 = 6a_3 - 6a_1 = 0$$

$$H_2 = 12a_4 - 2a_2 + 4a_0 = 0$$

$$H_3 = 20a_5 + 2a_3 + 4a_1 = 0$$

$$H_4 = 30a_6 + 6a_4 + 4a_2 - 8 = 0.$$

Assign $a_5=0$ and $a_6=0$, then there are 5 equations in 5 unknowns. In this case $a_0=1, a_1=0, a_2=2, a_3=0, a_4=0$. Also $a_5, a_6 \cdots$ are all zero. The particular solution is $y=e^{x^2}(2x^2+1)$.

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‡ Ince, Ordinary Differential Equations, Chapter VII.

** In each problem the rank of the matrix of the system (9) should be examined to determine the character of the solution. Finally the particular solution should be checked in (1).

If the expression for the $(k+1)$th term is written as a times the kth term, the resulting equation reduces to

$$n = ka + k - 1. \tag{1}$$

This means that for *any* a and *any* integer k, some binomial expansion exists (and is exhibited) whose $(k+1)$th term is a times its kth. That such a demand can so easily be met is scarcely intuitively evident. The condition for n to be integral is simply that ka be integral. For example, $a=3/2$, $k=6$, $n=14$, yield 2002 and 3003, the 6th and 7th coefficients of $(a+b)^{14}$.

Setting the $(k+2)$nd term $=b$ times the kth results in

$$n = \tfrac{1}{2}(2k - 1 \pm \sqrt{4bk^2 + 4bk + 1}). \tag{2}$$

The requirement that (1) and (2) hold simultaneously for arbitrary a, b and k exceeds by one the available degrees of freedom. In fact, (1) and (2) together are equivalent to

$$k = \frac{a+b}{a^2 - b} = \text{an integer.}$$

Certain special combinations lead to usable (integral) k. As a particular example, note that if $b=1$, k is integral when a is of the form $(c+1)/c$, c integral. This tells how to find an *integral* n such that the middle term of the expansion is $(c+1)/c$ times either adjacent term for any integral c.

Other similar properties are revealed by further very easy explorations. which the student may be encouraged to conduct on his own.

MATHEMATICS FOR BUSINESS STUDENTS

W. O. Buschman, Vanport Extension Center, Portland, Oregon

1. Introduction. For a number of years there has been considerable dissatisfaction with the traditional college mathematics program among those who are not majoring in the fields of mathematics and natural science. There have been many reports concerning the place of mathematics in General Education, but little has been written about the place of mathematics in the business curriculum.

In order to satisfy the mathematical needs of the business students some schools have arranged special courses for them. In other schools business students take the traditional mathematics courses or none at all. The nature of business itself makes this second type of program unsatisfactory, especially for students who are very weak in mathematics.

The purpose of this study is to consider the mathematical needs of the business student and to examine current trends and policies. Only higher education will be discussed here; this will mean only work beyond the traditional four-year high school. Only regular course work will be considered though it is admitted that outside or extracurricular activities are also important.

2. Mathematical requirements. The most noticeable thing about the mathematical requirements for business students, as shown by examination of various college catalogues, is the wide variation. In the schools of business the trend has been away from the traditional freshman courses: college algebra, trigonometry, and analytic geometry.* The integrated courses, when offered, generally do not differ much from the traditional sequence; the difference is more one of arrangement than of content.

Mathematics courses required of business students range all the way from none to calculus, and even to differential equations in one college. Most colleges require a program between these two extremes, including a course in mathematics of finance or mathematics of investment. College algebra is generally a prerequisite for these courses. Many colleges also require some work in elementary statistics.†

Generally where technical minors such as engineering, science, or similar fields are being combined with business the mathematical requirements are more rigorous and consist of the traditional type of courses plus a course in mathematics of investment. There is reason for this, as engineering, physics, *etc.* require some of the skills and concepts from trigonometry, analytic geometry, and calculus, even at the beginning level. A businessman dealing with people who are working in these fields will need to be acquainted with such topics himself.

In some universities the business program is entirely in the upper division or the graduate school; this plan appears to be part of the trend toward "general education" for all students during their first two years at a college or university. Specialization is thus postponed till after the second year. In these schools the lower division mathematical program of the business curricula will be the same as for "general education." For schools that do not have a "general education" program a special course should be provided in the lower division for all business students.

3. Aims of proposed course. The aims of an elementary mathematics course for all business students should be the development of a basic knowledge of and skills in arithmetic, algebra, and such computational devices as needed for their work. There is need of a working knowledge of processes dealing with loans, interest, investment, discounts, annuities, insurance of various types, and of some of the concepts of the development of these topics. Ability to analyze verbal problems and their solutions is of great importance and needs particular emphasis and development in the business curricula.

Most students entering the field of business have had little background in mathematics and nearly all will need to review fundamentals. For this reason the first course should require only a minimum of previous knowledge and skills

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* Monroe, Paul, Encyclopedia of Educational Research, Macmillan Co., New York, N. Y., 1950, p. 720.

† Hammond, Clela D., An Analysis of Mathematics Course Offered in Public Junior Colleges of California, Junior College Journal, Dec. 1949, p. 219.

in mathematics. At the same time, however, such a course must not be so elementary as to allow the student to lose interest, but should proceed to new topics as rapidly as possible. Elementary skills which may be lacking should not all be reviewed at the first of such a course as is the usual custom. These skills should be developed at the time they are extended and applied to more advanced topics. In this way the student will understand the need for these skills and the applications.

All students should be required to take a year course of 3 to 5 hours credit in mathematics; this should start with the fundamentals, including arithmetic concepts.‡ In most courses it has become customary to do all review and drill work first. This is very tedious for the better student, and he becomes bored. Review will be more effective if taken as needed, with the class going forward to new topics immediately.

4. General Outline of Proposed Course. 1. Development of algebraic notation and symbolism. This should be done through the use of equations and formulas taken from the field of business. Nearly all problems should be of the verbal type so that the formulas do not become drill exercises.

2. Percentage and its applications. Problems of cash discount, mark-ups, profit, loss, etc. can be used to develop facility with percentage. The use of index numbers in the statistical treatment of data will furnish additional applications or this may be taken later in the course thus providing a review of percentage.

3. Taxation. Income, property, and sales taxes, will furnish further applications of percentage and at the same time establish a familiarity with the processes of taxation which each student will meet sooner or later.

4. Graphical presentation of data. Graphs and formulas should be studied as a part of the development of the function concept. Problems of variation can readily be adapted to this aim.

5. Elementary concepts of statistics. A study of the measures of central tendency will provide further applications of formulas and give practice in the operations of arithmetic. If needed these operations can be reviewed. Considerable emphasis in the work of statistics should be on interpretation of the data. Statistical computations have little use or meaning without the interpretation of results. Most people will have many more occasions to interpret the meaning of statistical data than to compute it.

6. Time series. This will provide further applications of graphing and statistics, through a study of trends, seasonal as well as linear.

7. Further topics from statistics. Measures of central tendency will have developed the processes needed for a study of dispersion. Simple work with the binomial distribution and regression and correlation could be introduced if desired.

8. Progressions and series. These will be needed for further work in simple

‡ Newsom, C. V., A Course in College Mathematics for a Program of General Education, Mathematics Teacher, January 1949, pp. 19–24.

and compound interest. Applications of the progressions should be taken from verbal business problems as far as possible.

9. Computational aids such as logarithms and special tables will be needed before extensive work is done with interest and annuities. Calculators should be used if available.

10. Simple interest. This topic should be studied as an outgrowth of the arithmetic progressions. Emphasis should be on the development of the particular progression involved rather than direct applications of the formulas to develop the required concepts.

11. Compound interest. Amount and present value problems should be developed as an outgrowth of the geometric progression as indicated above for simple interest. The use of the compound interest and discount tables should be included.

12. Effective and equivalent rates of interest. Only through a study of these topics can the student gain an understanding of the high costs of deficit financing. These processes will give methods of comparing investments and loans. Applications should be taken from class situations, purchasing cars, houses, etc.

13. Annuities. For most students a treatment of annuities with simple data will be sufficient. Emphasis should be on the development of formulas and concepts of annuities. Some students will be reaching the limit of their ability to deal with abstractions and will have to deal with problems of annuities only as applications of formulas. Concepts should be developed as far as possible, however.

14. Amortization. This topic might well be introduced along with the one above to show the methods of liquidating debts. Sinking funds should also be included here. Comparisons of the methods of amortization and annuities should be stressed.

15. Bonds. Particular emphasis should be placed on the smaller government bonds with which the student is likely to come in contact. Bonding and bond investment on a larger scale can be used with local data such as municipal bonds to keep the problems within the area in which the student is familiar.

16. Depreciation. This is a topic about which all businesses are concerned, as it is important in taxation as well as for properly estimating profits and losses.

17. Building and Loan associations. The nature of this type of enterprise should be considered and compared with other types of investment.

18. Insurance. The various types of insurance are dealt with by almost everyone today. Life insurance, health insurance, car insurance, and many other types should all be considered as they are sure to be encountered sooner or later.

The general program outlined above could serve best for a year course to include all of the titles listed. It could be adjusted to either a "three-hour" or "five-hour" program; the material could be satisfactorily covered by a three hour course or more extensively in a four or five hour course. Parts of items 16, 17, and most of 15 could be omitted if the shorter course is desired. In the

earlier sections the review can be adjusted to the class, and most review omitted if the class does not need it.

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ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 931. *Proposed by H. D. Larsen, Albion College*

Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

E 932. *Proposed by Paul Erdös, University of Illinois*

(1) Prove that every positive integer not exceeding $n!$ is the sum of n or fewer distinct divisors of $n!$.

(2) Denote by $E(n) = 2^{a_2}3^{a_3} \cdots p^{a_p}$ the least common multiple of the positive integers not exceeding n. Prove that every positive integer not exceeding $E(n)$ is the sum of $a_2+a_3+\cdots+a_p$ or fewer distinct divisors of $E(n)$.

E 933. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let A', B', C' be the feet of the altitudes of a triangle ABC whose circumcircle and nine point circle are orthogonal. Show that (1) the areas of triangles ABC and $A'B'C'$ are equal, (2) $(AC')(BA')(CB') = (AB)(BC)(CA)/2$. (3) Construct the triangle ABC given side BC and the position of A' on BC.

E 934. *Proposed by C. O. Oakley, Haverford College*

Let C_1 and C_2 be two arbitrarily given curves whose parametric equations are $x_1 = x_1(t)$, $y_1 = y_1(t)$ and $x_2 = x_2(t)$, $y_2 = y_2(t)$. Show that there exists a third curve C, $x = x(t)$, $y = y(t)$, with the "focal" property that, if P_1, P_2, and P are corresponding points on C_1, C_2, and C respectively, then P_1P and P_2P make equal angles with the tangent to C at P.

E 935. *Proposed by C. D. Olds, San Jose State College*

Give an example of a function which is discontinuous in an everywhere dense set, and which is also differentiable in an everywhere dense set.

SOLUTIONS

An Unsolved Problem

E 534 [1942, 475, 1943, 261]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Solution by H. D. Grossman, New York, N. Y. The only values of $n<10$, for which $n!+1$ is a perfect square, are 4, 5, and 7. Then let $n \geqq 10$ and let all letters in the discussion be positive integers.

If $n!+1 = k^2$, then $n! = (k+1)(k-1)$. Let the integers $k+1$, $k-1$ be r, s in unknown order. Since r, s differ by 2 and their product, $n!$, is even, they must both be even. Also, they can have no common even factor but 2 and no common odd factor at all (besides unity).

Then 2 divides only one of them, say r, while (4) (6) (8) (10) $\cdots$ divides the other. Then (3) (5) (9) also divides s. At most r is divisible by 2 and by those odd primes whose doubles exceed n. These restrictions make r so much smaller than s that r and s cannot possibly differ by only 2.

In the following table the greatest possible chain of factors of r is in the left column. Each of these is exceeded by the corresponding factor of s in the right column.

r		s
2		(3)(4)
7		(5)(6)
$6t-1$	$(t>1)$	$(6t-2)(6t-3)$
$6t+1$	$(t>1)$	$6t(6t-4)$

Whereas the product of factors in the right column is much greater than that in the left;

And whereas, as n increases, every odd prime $6t \pm 1$ in the left column must be permanently transferred to the right column as soon as n reaches $2(6t \pm 1)$;

And whereas all composite $6t \pm 1$ do not even fall initially in the left column;

And whereas no integer is repeated anywhere in the table;

Be it hereby resolved that $n!+1$ can never be a perfect square if $n \geqq 10$.

A Dissected Cube

E 897 [1950, 34]. *Proposed by W. R. Ransom, Tufts College*

A cube is divided into equal parts by the plane of a hexagon whose vertices are midpoints of its edges. In what ratio is a cube divided by a parallel plane that divides an edge in the ratio $a:(1-a)$?

Solution by C. M. Sandwick, Easton High School, Easton, Pa. Imagine an observer at a point outside a unit cube in line with the diagonal r which is perpendicular to the cutting plane. Imagine that the plane passes through the cube while approaching the observer. The plane travels a distance $r/3$ from the time that it passes the most remote vertex until it contains the next three vertices simultaneously and during this interval it cuts three edges. The plane travels a second distance $r/3$ until it again contains three vertices simultaneously, during which time it intersects six edges. Finally the plane travels the distance $r/3$ until it leaves the cube as it passes the last vertex, cutting the three remaining edges as it traverses the final interval. In each of the three stages assume that a varies from 0 to 1 as the plane approaches the observer. Let v denote that part of the volume which is on the side of the plane away from the observer. In a unit cube $r = \sqrt{3}$. In each stage, since the projection of an intersected edge on r is $\sqrt{3}/3$, we see that $dr = da/\sqrt{3}$. Let K denote the area of the intersection of the plane and the cube.

In the first stage, as a increases from 0 to 1, v increases from 0 to 1/6, and $K = a^2\sqrt{3}/2$. Integrating $K\,dr$, we obtain $v = a^3/6$. Therefore, in this region,

$$v:(1 - v) = a^3:(6 - a^3).$$

In the second stage, as a increases from 0 to 1, v increases from 1/6 to 5/6, and it is easy to show that $K = \sqrt{3}(1/2 + a - a^2)$. Integrating $K\,dr$ we then find that $v = (1 + 3a + 3a^2 - 2a^3)/6$. Hence

$$v:(1 - v) = (1 + 3a + 3a^2 - 2a^3):(5 - 3a - 3a^2 + 2a^3).$$

In the third stage, as a increases from 0 to 1, v increases from 5/6 to 1, and $K = (1-a)^2\sqrt{3}/2$. Hence $v = [6 - (1-a)^3]/6$, and

$$v:(1 - v) = [6 - (1 - a)^3]:(1 - a)^3.$$

Also solved, for the second stage, by W. G. Brady, D. H. Browne, Roger

Lessard, L. A. Ringenberg, Nathan Schwid, C. W. Trigg, and the proposer.

Trigg further noted that, in the second stage, the surface area of the cube is divided in the ratio $(1+2a):(3-2a)$, which is also the ratio of the edges (and portions of edges) of the cube contained in the two pieces. The perimeter of the hexagonal section of the unit cube is $3\sqrt{2}$.

Construction of an Equilateral Triangle

E 899 [1950, 35]. *Proposed by E. R. Bowersox, Chicago, Illinois*

Construct an equilateral triangle given the distances of a point in its plane from the three vertices.

Solution by C. W. Trigg, Los Angeles City College. CONSTRUCTION. We take the distances $a \geqq b \geqq c$ from the point P. On $PD=a$ construct an equilateral triangle PDA and triangles PDC and PDC' with $PC=c=PC'$ and $DC=b=DC'$. On the other side of AC from D construct the required equilateral triangle ABC. Similarly, construct equilateral triangle $AB'C'$. Clearly there are two ($a<b+c$ and a, b, c not all equal), one ($a=b+c$ or $a=b=c$) or no ($a>b+c$) solutions.

PROOF. Draw PB and PB'. By construction, $PA=DA=a$, $CA=BA=x$, and $\sphericalangle DAC+\sphericalangle CAP=60^\circ=\sphericalangle CAP+\sphericalangle PAB$, so $\sphericalangle DAC=\sphericalangle PAB$. Hence, triangles DAC and PAB are congruent, so $PB=DC=b$. In like manner, $PB'=b$.

Also solved by Norman Anning, J. H. Braun, D. H. Browne, W. E. Buker, W. B. Clarke, L. P. Coggin, William Douglas, R. R. Eddy, B. H. Gundlach, Frank Herlihy, H. K. Humphrey, John Jones, Jr., L. M. Kelly, M. S. Klamkin, Roger Lessard, Prasert Na Nagara, C. C. Oursler, A. P. Rhodes, L. A. Ringenberg, Joseph Rosenbaum, N. C. Scholomiti, Alfred Sylvester, and the proposer.

Editorial Note. Most of the solutions were based upon either a stretch-rotation or intersecting circles of Apollonius, and could easily be generalized to yield the construction of a triangle of assigned species having its vertices at given distances from a given point. This generalized problem is sometimes referred to as *Carnot's problem*. For a solution see, *e.g.*, N. A. Court, *College Geometry*, sec. 56, p. 49. For a discussion and history of Carnot's problem see problem 2761 [1920, 138]. Here it is shown, for the given problem, that

$$x^2 = (a^2 + b^2 + c^2)/2 \pm 2\sqrt{3}\,K,$$

where K is the area of triangle PDC. Also see problems 514 [1918, 21] and 3904 [1941, 75]. For a generalization in a different direction see *Proc. Ed. Math. Soc.*, vol. 27 (1909), pp. 70–75 and vol. 28 (1910), pp. 95–120. Trigg gave several references to problems in *School Science and Mathematics* where the configuration of this problem has received some study. It is intimately related to the identity connecting the six distances of four points in a plane.

Some Cospherical Circles

E 900 [1950, 35]. *Proposed by N. A. Court, University of Oklahoma*

If four spheres may be divided into pairs so that the circle common to one pair is cospherical with the circle common to the second pair, then the circle determined by any two of the given spheres is cospherical with the circle determined by the remaining two spheres.

Solution by L. M. Kelly, Michigan State College. Let the spheres be (A), (B), (C), (D), and suppose the circles of intersection of (A) and (B) and of (C) and (D) are on a sphere (S). The center of (S) is on the line of centers AB, and also on the line of centers CD. Thus A, B, C, D are coplanar. The spheres (A), (B), (S) have a common radical plane, as have (C), (D), (S). These two radical planes intersect in a line perpendicular, at P say, to the plane of centers of the four spheres. The powers of P relative to the four spheres are all equal. Let X_1, X_2, and Y_1, Y_2 be the points where the circles of intersection of (A), (C) and of (B), (D) cut the plane of centers. Then $(PX_1)(PX_2) = (PY_1)(PY_2)$, and the points X_1, X_2, Y_1, Y_2 are concyclic. It follows at once that the two circles are cospherical. A similar argument may be applied to the remaining pair of circles.

Also solved by Roger Lessard and the proposer.

Editorial Note. The analogous theorem in the plane is also true, and reads as follows: If four circles may be divided into two pairs so that the intersections of one pair are concyclic with those of the second pair, then the intersections of any two of the circles are concyclic with those of the remaining two circles.

Bonus Problem

E 901 [1950, 113]. *Proposed by Leo Moser, University of Manitoba*

A company offers its 350 employees a bonus of \$10.00 to each male and \$8.25 to each female. All the females accept, but a certain percentage of the males refuse to accept. Knowing what this percentage is, one can deduce the total amount paid out. What is the total amount paid to the women?

Solution by Bart Park, Michigan College of Mining and Technology. Let m be the number of men and let r be the fraction of men refusing a bonus. Then the amount paid out is given by

$$T = 10m(1 - r) + (350 - m)(8.25),$$

which will be independent of m if and only if $r = 0.175$. We find that $T = 2887.50$ dollars. Both m and $0.175m$ must be integers, with $m < 350$. The following values of m satisfy these requirements: 40, 80, 120, 160, 200, 240, 280, 320. There are therefore eight possible amounts, given by $(350 - m)(8.25)$, which may be paid to the women.

Also solved by W. E. Buker, M. S. Klamkin, C. S. Ogilvy, W. O. Pennell, J. V. Pennington, A. P. Rhodes, L. A. Ringenberg, C. W. Trigg, G. W. Walker, and the proposer.

Editorial Note. As was pointed out by some solvers, the solution could have been made unique by paying a more judicious bonus to each woman, say \$8.15 or \$8.35, instead of \$8.25. There are other values of r for which T has a single value, but here finding T would involve finding m. Walker found all such solutions in addition to those given above.

Rational Points on a Sphere

E 902 [1950, 113]. *Proposed by R. V. Andree's class in Solid Analytics, University of Oklahoma*

A similar argument to that used in E 832 [1949, 407] may be used to "prove" the following theorem: "If a sphere has a center with at least one irrational coordinate, then there are at most three points on the sphere with rational coordinates." Give a counter example to show that this theorem is not true and discuss the point at which the "proof" breaks down.

Solution by L. A. Ringenberg, Eastern Illinois State College. A counter example is any sphere in the family $x^2+y^2+z^2+Az=1$, where the parameter A is any irrational number. Four rational points on this sphere are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$. The proof of E 832 cannot be extended to the case of a sphere; for while it is true that if three points lie on a circle, then they lie on only one circle, it is not true that if four points lie on a sphere, then they lie on only one sphere.

Also solved by Paul Baird, Vern Hoggatt, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, and the proposers.

Editorial Note. Of course we do have the true theorem: "If a sphere has a center with at least one irrational coordinate, then there cannot be four *non-coplanar* points on the sphere with rational coordinates."

On the other hand, it is easy to show that there exist spheres of arbitrary size, having non-rational centers, and passing through infinitely many (co-planar) rational points. To accomplish this we may proceed as follows. It is known that $x^2+y^2=1$ is satisfied by infinitely many pairs of rational values for x and y (see, *e.g.*, H. N. Wright, *First Course in Theory of Numbers*, p. 93), and therefore the circle C: $x^2+y^2=r^2$, where r is rational, passes through infinitely many rational points. Any sphere through C thus passes through infinitely many rational points. For the center of the sphere we may choose any non-rational point on the z-axis.

Even Integers as Sum of Two Abundant Numbers

E 903 [1950, 113]. *Proposed by F. A. E. Pirani, Carnegie Institute of Technology*

Show that every even integer greater than 46 can be expressed as the sum of two abundant numbers. (See E 848 [1949, 478].)

I. *Solution by Leo Moser, University of North Carolina.* An explicit representation of every even integer greater than 46 as the sum of two abundant num-

bers is given by

$$48 + 6k = 30 + (3 + k)6,$$
$$50 + 6k = 20 + (5 + k)6,$$
$$52 + 6k = 40 + (2 + k)6,$$
$$k = 0, 1, 2, \cdots.$$

II. *Solution by John Selfridge, University of Washington.* Any even number $m > 46$ can be written as $6k$, $6k+2$, or $6k+4$, where $k \geqq 8$. We have

$$6k = 6(k - 2) + 12,$$
$$6k + 2 = 6(k - 3) + 20,$$
$$6k + 4 = 6(k - 6) + 40.$$

Since 20, 40, and $6n$ $(n > 1)$ include all abundant numbers less than 46 it follows that 46, 34, 28, 26, and all even numbers less than 24 are the only even numbers not expressible as the sum of two abundant numbers. However, allowing the summands 6, 28, and 0 we may say: Any even number greater than 22 is non-deficient or can be expressed as the sum of two non-deficient numbers.

Also solved by Roger Lessard, C. W. Trigg, and the proposer.

Trigg further showed that every even integer greater than $12M+26$ is expressible as the sum of M abundant numbers. He also observed that 24, 36, 48, 60, 66, 84 are the smallest integers which can be expressed as the sum of two abundant numbers in 1, 2, 3, 4, 5, 6 ways, respectively.

A Cubic Locus of Centers of Curvature

E 904 [1950, 113]. *Proposed by C. S. Ogilvy, Columbia University*

Find the equation of the locus of centers of curvature of $y = x^n$ at the point (1, 1) as n varies continuously from $-\infty$, $+\infty$.

Solution by S. T. Thompson, Tacoma, Washington. Using the standard formulas for center of curvature one obtains, as parametric equations for the locus,

$$x = 1 - (n^2 + 1)/(n - 1), \qquad y = 1 + (n^2 + 1)/n(n - 1).$$

Eliminating the parameter by means of the relation

$$n = -(x - 1)/(y - 1),$$

one obtains the cartesian equation

$$(1 - x)^2(2 - y) + (1 - y)^2(2 - x) = 0.$$

This represents a unicursal acnodal cubic with its isolated double point at U:(1, 1). The point U, however, is not part of the required locus, for it corresponds to $n = \pm i$, and n is restricted to real values only. The cubic is tripartite having symmetry in the line $y = x$ and having $x = 2$, $y = 2$, $x = -y$ as asymptotes.

The branches correspond to the parametric ranges $-\infty < n < 0$, $0 < n < 1$, $1 < n < \infty$. The first branch is in the first quadrant, passing through the point (2, 2) and asymptotic to both $x = 2$ and $y = 2$. The second branch is in the fourth quadrant having $x = 2$ and $x = -y$ as asymptotes and lying within the infinite sector bounded by these asymptotes. The third branch is the reflection of the second branch in the line $x = y$. The curve is not found in the third quadrant nor in the infinite strips

$$-2\sqrt{2} - 1 < x < 2\sqrt{2} - 1 \quad \text{and} \quad -2\sqrt{2} - 1 < y < 2\sqrt{2} - 1.$$

The parameter n has simple geometric significance. Let P be any point on the curve and let UX be parallel to the positive x-axis. Then

$$n = -\cot (XUP).$$

If UX is chosen as a polar axis, the polar equation of the curve can be written as

$$r \sin \theta \cos \theta(\sin \theta + \cos \theta) = 0.$$

Also solved by W. G. Brady, P. L. Chessin, Ragnar Dybvik, Kurt Eisemann, Frank Herlihy, C. J. Kaufman, M. S. Klamkin, Roger Lessard, J. E. Morgan, A. Sisk, C. G. Solky, O. E. Stanaitis, John Torian, and the proposer.

A Property of the Orthocenter of a Triangle

E 905 [1950, 113]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1 and A_2, B_1 and B_2, C_1 and C_2 be the points of intersection with the sides BC, CA, AB of a triangle ABC of the polars of a point P with respect to the escribed circles (I_b) and (I_c), (I_c) and (I_a), (I_a) and (I_b). For what position of P will the points A_1 and A_2, B_1, and B_2, C_1 and C_2 be, simultaneously, symmetric with respect to the midpoints of BC, CA, AB?

Solution by the Proposer. This problems leads to an interesting and probably new property of the orthocenter H of a triangle.

LEMMA. *A necessary and sufficient condition for the polars of two points A_1 and A_2, taken on the side BC of a triangle ABC, with respect to the escribed circles (I_b) and (I_c), to intersect on the altitude AA' through vertex A, is that A_1 and A_2 be symmetric with respect to the midpoint of BC.*

Let A_1 and A_2 be symmetric with respect to the midpoint of BC, and let (I_b) and (I_c) be tangent to BC at D_b and D_c respectively. Let the polar of A_1 with respect to (I_b) cut D_cI_c in M and the polar of A_2 with respect to (I_c) cut D_bI_b in N. Then, from the pairs of similar right triangles $D_bI_bA_1$, D_cD_bM and $D_cI_cA_2$, D_bD_cN, and the fact that $A_1D_b = A_2D_c$, we have

$$ND_b : MD_c = D_bI_b : D_cI_c = AI_b : AI_c.$$

The point of intersection, A'', of the polars D_cN and D_bM thus lies on a parallel to D_bI_b and D_cI_c through the external center of similitude, A, of (I_b) and (I_c). That is, A'' lies on the altitude AA'. Conversely, by reversing the above argument, it may be shown that if A'' is on the altitude AA', then A_1 and A_2 are symmetric with respect to the midpoint of BC. This proves the lemma.

It now follows that the altitude line AA' is the locus of a point P whose polars with respect to (I_b) and (I_c) will pass through a pair of isotomic points A_1, A_2 of BC. Similarly, the altitude lines BB', CC' of triangle ABC are the loci of points P for which the polars with respect to (I_c) and (I_a), (I_a) and (I_b) cut CA and AB in pairs of isotomic points. In order that A_1 and A_2, B_1 and B_2 C_1 and C_2 be, simultaneously, isotomic points on BC, CA, AB, it is necessary and sufficient that P coincide with the orthocenter H of ABC.

Also solved by Roger Lessard.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4410. *Proposed by C. D. Olds, San Jose State College, California*

Show that, for every positive integer n, there is a polynomial $f(x)$ of degree n and a related polynomial $\phi(x)$ which satisfy

$$[f(x)]^2 - 1 = (x^2 - 1)[\phi(x)]^2.$$

4411. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Show that if α is a real number >0, then for any given $\epsilon>0$ and any integer $k>0$ there can be found a number $\delta=\delta(\epsilon)>0$ and an integer $N\geqq k$ such that, if

$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = \Delta x_1' + \Delta x_2' + \cdots + \Delta x_N' = \alpha;$$
$$S_k = \sum \Delta x_{\nu_1} \cdots \Delta x_{\nu_k}, \qquad S_k' = \sum \Delta x_{\nu_1}' \cdots \Delta x_{\nu_k}',$$

then

$$|S_k - S_k'| < \epsilon$$

whenever $0<\Delta x_i<\delta$, $0<\Delta x_i'<\delta$, where S_k and S_k' are extended over all possible terms with distinct subscripts $\nu_1, \cdots, \nu_k$.

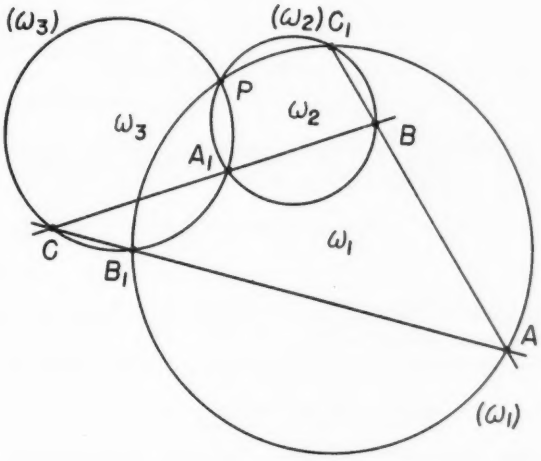
SOLUTIONS

Circumcircles of a Family of Similar Triangles

4301 [1948, 369]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider similar triangles ABC, the sides BC, CA, AB of which pass through the fixed points A_1, B_1, C_1. (1) The locus of the circumcenters O of these triangles is a circle. (2) The circumcircles (O) are orthogonal to a fixed circle. (3) When the points A_1, B_1, C_1 are collinear, the envelope of the circles (O) is a cardioid. What is this envelope otherwise?

Solution by W. E. Byrne, Lexington, Virginia. We prefer the following statement of the problem:* *Given three circles* $AB_1C_1 \equiv (\omega_1)$, $A_1BC_1 \equiv (\omega_2)$, $A_1B_1C \equiv (\omega_3)$, *having the fixed point* P *in common. If* A, B, C *vary subject to the condition that* BC, CA, AB *always pass through the distinct fixed points* A_1, B_1, C_1, *show that the locus of the circumcenters* Γ *of* ABC *is a circle, show that the circumcircles* (Γ) *are orthogonal to a fixed circle if* A_1, B_1, C_1 *are not collinear, and find the envelope of the circumcircles* (Γ). *Discuss also the special case for which* A_1, B_1, C_1 *are collinear.*



It is convenient to use a system of complex coördinates $z = x + iy$, $\bar{z} = x - iy$. Let us choose as a unit reference circle the circumcircle of triangle $\omega_1\omega_2\omega_3$,† of center $\omega(z=0)$, with the point P $(z=a)$ on the positive half of the real axis. If t_1, t_2, t_3 denote the points ω_1, ω_2, ω_3, we have as the equation of (ω_k), $k = 1, 2, 3$,

$$z\bar{z} - \bar{t}_k z - t_k \bar{z} = a^2 - \bar{t}_k a - t_k a.$$

* As originally stated the problem gives part of the loci here obtained, due to the fact that only portions of the circles (ω_k) are involved.

† ω_1, ω_2, ω_3 are not collinear since otherwise the circles (ω_k) would have an axis of symmetry and A_1, B_1, C_1 would coincide in a point symmetric to P.

The coördinates of A_1, B_1, C_1 are

$$z_{A_1} = t_2 + t_3 - at_2t_3, \qquad z_{B_1} = t_3 + t_1 - at_3t_1,$$
$$z_{C_1} = t_1 + t_2 - at_1t_2.$$

If the position of A is fixed by $z_A = t_1 + \mu(a - t_1)$, $|\mu| = 1$, then it follows that

$$z_B = t_2 + \mu(a - t_2), \qquad z_C = t_3 + \mu(a - t_3).$$

The equations of the perpendicular bisectors of AB and BC, are, respectively,

$$z + \mu t_1 t_2 \bar{z} = a(\mu + t_1 t_2), \qquad z + \mu t_2 t_3 \bar{z} = a(\mu + t_2 t_3),$$

and hence the circumcenter Γ of triangle ABC has the affix $z = a\mu$, and we conclude that the locus of Γ is the entire circle $z\bar{z} = a^2$ of center ω and radius a.

The equation of the circumcircle (Γ) is

$$z\bar{z} - \mu a\bar{z} - \bar{\mu} a z + a^2 = 2 - \mu - \bar{\mu}.$$

The condition that (Γ) be orthogonal to the circle having equation $z\bar{z} - b\bar{z} - \bar{b}z + c = 0$ is $ba\bar{\mu} + \bar{b}a\mu = a^2 + \mu + \bar{\mu} - 2 + c$, or

$$\mu(1 - a\bar{b}) + \bar{\mu}(1 - ab) + a^2 - 2 + c = 0.$$

If we choose $b = \bar{b} = 1/a$, $c = 2 - a^2$, we have a fixed circle

$$z\bar{z} - (z + \bar{z})/a + 2 - a^2 = 0,$$

of center $z = 1/a$ and radius $|1 - a^2|/a$, orthogonal to the circles (Γ).

A_1, B_1, C_1 are collinear if and only if

$$\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{t_1 t_2 t_3}(a^2 - 1) = 0.$$

In this case the circle orthogonal to the circles (Γ) reduces to a null circle (P) and P is on the circumcircle of $\omega_1\omega_2\omega_3$.

In both cases the envelope of (Γ) is given by the discriminant of

$$(1 - a\bar{z})\mu^2 + (z\bar{z} + a^2 - 2)\mu + 1 - az = 0,$$

where use is made of the fact that $\mu\bar{\mu} = 1$, that is by the equation

$$(z\bar{z} + a^2 - 2)^2 - 4(1 - a\bar{z})(1 - az) = 0.$$

If the point P is chosen as the pole in a system of polar coördinates, with ωP serving as the initial line, the envelope appears as the graph of $r = 2(1 - a\cos\theta)$, which is a limaçon. It is a cardioid for $a = 1$.

Also solved by Robert Bouvaist and the Proposer.

Editorial Note. The Proposer's solution uses pure geometry. The first result follows very easily through the use of the special triangle $A_0B_0C_0$ determined by the perpendiculars drawn through A_1, B_1, C_1 to PA_1, PB_1, PC_1. Let Γ_0 be the

circumcenter of $A_0B_0C_0$. Since P is the center of similitude of the triangles $A_0B_0C_0$ and ABC, triangles $P\Gamma_0A_0$ and $P\Gamma A$ are similar, and so $P\Gamma_0\Gamma$ and PA_0A are similar. But the angle at A of triangle PA_0A is a right angle, and hence Γ describes the circumference of diameter $P\Gamma_0$.

Sequence of Integers No One of Which Divides Another

4332 [1949, 112]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Prove that from the sequence a_i+a_j, $i=1, 2, \cdots$, $j=1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

Solution by the Proposer. We distinguish two cases.

Case I. In every infinite subsequence, $b_1, b_2, \cdots$, of the a's there exists at least one sum b_i+b_j and a corresponding infinite subsequence $b_{i_1}, b_{i_2}, \cdots$ satisfying

$$2b_{i_k} \not\equiv 0 \pmod{(b_i + b_j)}.$$

It follows that we can select infinitely many of them all in the same residue class. In other words, there exists an $a_{i_1}+a_{j_1}$ and infinitely many a's (for convenience we denote them by $a_1^{(2)}, a_2^{(2)}, \cdots$) such that

$$(1) \qquad 2a_k^{(2)} \not\equiv 0 \pmod{(a_i + a_j)}, \qquad a_1^{(2)} \equiv a_2^{(2)} \equiv \cdots \pmod{(a_i + a_j)}.$$

In the same way we see that there again exists $a_{i_2}^{(2)}+a_{j_2}^{(2)}$ such that there is an infinite subsequence of the $a_i^{(2)}$'s (say $a_1^{(3)}, \cdots$) satisfying

$$2a_k^{(3)} \not\equiv 0 \pmod{(a_{i_2}^{(2)} + a_{j_2}^{(2)})}, \qquad a_1^{(3)} \equiv a_2^{(3)} \equiv \cdots \pmod{(a_{i_2}^{(2)} + a_{j_2}^{(2)})}.$$

From (1) it follows clearly that

$$a_{i_2}^{(2)} + a_{j_2}^{(2)} \equiv 2a_{i_2}^{(2)} \not\equiv 0 \pmod{(a_i + a_j)}.$$

Similarly we obtain

$$a_{i_3}^{(3)} + a_{j_3}^{(3)}, \ a_{i_4}^{(4)} + a_{j_4}^{(4)}, \cdots,$$

and clearly no one divides any other.

Case II. There exists a subsequence, $b_1, b_2, \cdots$, of the a's such that to every b_i+b_j there are only a finite number of b_k's for which $2b_k \not\equiv 0 \pmod{(b_i+b_j)}$. Hence for all n sufficiently large

$$2b_n \equiv 0 \pmod{(b_i + b_j)}.$$

Consider then the sequence $b_1+b_{n_1}, b_1+b_{n_2}, \cdots$, where $n_1<n_2<\cdots$ tend to infinity fast enough so that

$$2b_{n_k} \equiv 0 \pmod{(b_1 + b_{n_i})}, \qquad i < k.$$

Further, since $0<2b_1<b_1+b_{n_i}$,

$$2b_1 \not\equiv 0 \pmod{(b_1 + b_{n_i})}.$$

Thus

$$2(b_1 + b_{n_k}) \equiv 2b_1 \pmod{(b_1 + b_{n_i})}$$

and clearly no one divides another.

The Bennett Point of a Quadrangle

4334 [1949, 112]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the feet of the six perpendiculars from the Bennett point on the sides of the complete quadrangle lie on a conic.

Solution by the Proposer. This will follow if the locus of the feet of perpendiculars from the Bennett point* on the asymptotes of conics through four points is proved to be a conic.

To find the degree of the locus when the point is general, consider the points it has in common with the line at infinity. These will occur when an asymptote passes through a circular point or is the line at infinity itself. There are two conics one through each of the circular points and there are two touching the line at infinity. The curve is therefore a quartic.

In the particular case when the point is the intersection of the tangents at the circular points to the respective conics through these points, that is the Bennett point of the quadrangle, these tangents are evidently part of the locus. The remainder is therefore a conic.

It may be remarked that this conic has asymptotes perpendicular to those of the conic which is the locus of centers of the conics through the four points—since it passes through the harmonic conjugates with respect to the circular points of the points of contact of the two parabolas with the line at infinity—and passes through the Bennett point and the center of the unique rectangular hyperbola through the four points.

Fibonacci Numbers

4341 [1949, 268] *Proposed by D. H. Browne, Buffalo, New York*

The sequence $\{a\} = 3, 7, 47, 2207, 4870847, \cdots$, used for the determination of primality of the Mersenne numbers, is usually defined by

$$a_{n+1} = a_n^2 - 2.$$

Show that it may also be defined by

$$a_k = f_{2^{k+1}}/f_{2^k},$$

where the f's are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, $\cdots$.

* The Bennett point is the point from which the circumcircles of the four points three at a time look the same size. See Morley and Morley, *Inversive Geometry*, p. 254, ex. 7, also p. 250.

Solution by John Todd, East Molesey, Surrey, England. It is well known that $f_n=(r^n-s^n)/\sqrt{5}$, where r and s $(-r^{-1})$ are the roots of $x^2-x-1=0$. Hence, writing $a=2^n$, we have

$$\frac{f_{4a}}{f_{2a}}-\left(\frac{f_{2a}}{f_a}\right)^2+2=\frac{r^{4a}-s^{4a}}{r^{2a}-s^{2a}}-\left(\frac{r^{2a}-s^{2a}}{r^a-s^a}\right)^2+2$$

$$=r^{2a}+s^{2a}-(r^a+s^a)^2+2=-2(rs)^a+2=0.$$

Therefore $f_{2^{k+1}}/f_{2^k}$ satisfies the same recurrence relation as a_k and since $a_1=3=f_4/f_2$ the desired result is proved.

Also solved by A. C. Aitken, Murray Barbour, W. R. Cowell, J. S. Frame, R. Goormaghtigh, Hansraj Gupta, B. F. Hadnot, R. N. Haskell, Vern Hoggatt, Free Jamison, Irving Kaplansky, M. S. Klamkin, N. D. Lane, Roger Lessard, D. C. B. Marsh, Leo Moser, C. D. Olds, S. T. Parker, C. F. Pinzka, L. B. Rall, Joseph Rosenbaum, Alex Rosenberg, Azriel Rosenfeld, N. C. Scholomiti, Robert Steinberg, J. D. Swift, and the Proposer.

Editorial Note. Several solvers point out that a_k may be given also as

$$a_k = f_{2^k+1}+f_{2^k-1}, \tag{1}$$

$$a_k = g_{2^k} \tag{2}$$

where $\{g\}$ is defined by $g_{n+2}=g_{n+1}+g_n$; $g_1=1$, $g_2=3$.

Cowell, Goormaghtigh and Rosenberg consider the more general quadratic $x^2-bx+c=0$, whose roots are r and s. If $u_n=(r^n-s^n)/(r-s)$, we have

$$u_{2^{n+1}}/u_{2^n}=r^{2^n}+s^{2^n},$$

whence, if the left member is represented by a_n, it is easily verified that

$$a_{n+1}=a_n^2-2c^{2^k}.$$

Further generalization and applications may be found in Kaplansky, *Lucas's Test for Mersenne Numbers*, This MONTHLY [1945, 188].

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

You Can't Win. By E. E. Blanche. Washington. Public Affairs Press, 1949. 155 pages. $2.00.

This book is concerned with various aspects of gambling in this country. It consists largely of descriptive material and does not attempt very profound analyses of games of chance. The author seems thoroughly convinced of the "pernicious nature" of gambling and that all professional gamblers are essentially dishonest. In some cases the title assertion is supported by accounts of the classical mathematical analyses, but no very complete study is made of any particular game.

In so far as the reviewer noted, odds and probabilities quoted are correct. The reviewer is somewhat dubious concerning an argument which seems to maintain that the second card in a Bingo game is a much poorer investment than the first one. He also feels that the emphasis on confidence men, carnival games and dishonest practices serves to decrease the potential interest of the book to the mathematician. The reader who wants some information concerning the rules of various games and the underlying mathematics and who is somewhat predisposed to the author's general attitude should find this an interesting work.

E. M. BEESLEY

Probability Theory for Statistical Methods. By F. N. DAVID. Cambridge University Press, 1949. 10+230 pages. $3.50.

A leading British statistician has written this book to serve as an elementary textbook on probability theory for students who wish to study statistics. Actually, however, this book discusses a number of topics in the statistical theory of estimation. Two introductory chapters are followed by four chapters on the binomial distribution. Then follow chapters entitled as follows: VII Probabilities a posteriori, Confidence limits, VIII Simple genetical applications, IX Multinomial theorem and simple combinatorial analysis, X Random variables. Elementary laws and theorems, XI Moments of sampling distributions, XII Random variables. Inequalities. Laws of large numbers. Lexis theory, XIII Simple estimation. Markoff theorem on least squares, XIV Further applications of the Markoff theorem. Chapters XV, XVI, XVII deal with characteristic functions.

The author of the book writes with an enthusiasm for her subject which is bound to communicate itself to the student. There are many lively and meaningful examples, drawn from statistical applications, which should convince the

student that the field of application of probability theory goes far beyond problems about gambling or about very dull urns which somehow, in an unexplained manner, always manage to contain balls of several colors. The mathematical argument is kept within the reach of a student with a knowledge of what in the United States is called advanced calculus, or perhaps even less. There is a great lack of good books on probability with these qualities.

The chief fault of the book is that no distinction is made between probability theory and the theory of statistical estimation. The arrangement of the chapters and certain remarks of the author heighten the confusion which must arise in the mind of the student. For example, on page 71 the author precedes what is meant to be (and actually is) the proof of an innocuous result known as Bayes' theorem with the dire warning that "the modern point of view is that, strictly speaking, the theorem is wholly fallacious except under very restricted conditions." (She is obviously referring to the use of Bayes' theorem for statistical estimation.) On page 72 it is said that, "unless $P\{E_t\} \cdots$ is known the formula cannot be valid." Obviously the validity of a formula is not a function of our knowledge.

There are a number of mathematical inadequacies, of which we cite a few. The definition of probability on page 13 is the one hallowed by use in many textbooks, but unsatisfactory nevertheless. On page 22 the author states that "the favorite problem of the probability of drawing balls from an urn is incalculable unless it is assumed that all balls have an equal probability of being drawn. . . ." Why? The definition of a continuous chance variable on page 112 is really that of an absolutely continuous chance variable. The student should not be misled into thinking that every continuous monotonic function is the integral of its derivative. In the proof of Liapounoff's theorem on page 217 there is no need to bound the variances above or to bound the absolute third moment below. No doubt these inadequacies will be corrected in subsequent editions of the book.

J. WOLFOWITZ

Reissner Anniversary Volume: Contributions to Applied Mechanics. Edited by the Staff of the Department of Aeronautical Engineering and Applied Mechanics of the Polytechnic Institute of Brooklyn, J. W. Edwards, Ann Arbor, Michigan, 1949. VIII+493 pp. $6.50.

This volume has been dedicated to Professor Hans J. Reissner of the Polytechnic Institute of Brooklyn on the occasion of his seventy-fifth birthday (Jan. 18, 1949). In addition to a short biography of Professor Reissner and a list of his scientific publications from 1899 to 1948 (92 entries) the volume contains 32 papers contributed by friends and former students of Professor Reissner. Obviously, the space available for this revue does not permit the reviewer to do justice to the individual contributions: a list of the authors and the fields of their contributions will have to suffice. There are 5 papers on aerodynamics (by Bergman, Harrington and Libby, Lew, Riabouchinsky, Tollmien), 5 papers on dynamics (by Goland, Lieber and Hamilton, Oldenburger, Williams, Yuan

and Morduchow), 7 papers on elasticity (by Donnell, Friedrichs, Gran Olsson, E. Reissner, Schleusner, Schnadel, Stoker), 5 papers on aircraft structures (by Hoff, Salerno, Liebowitz, Boley, Nardo), 2 papers on electricity (by Foster, Rüdenberg), 3 papers on mathematical methods (by Geiringer, Grammel, Weinstein), 3 papers on plasticity (by v. Mises, Nadai, Odqvist), and 2 papers on propulsion (by v. Kármán, Torda).

W. PRAGER

Theory and Application of $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$. (Part I. Methods of Computation.) By J. B. Rosser. New York, Mapleton House, 1948. 4+192 pages $8.00.

Part II of the monograph *Theory and Application of* $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy\int_0^y e^{-x^2}dx$ has not appeared in full as it contains information which is restricted. All secrecy on the contents of Part I has been lifted and it has been reproduced by arrangement with the Office of Technical Services Department of Commerce. A large share of Part I which is entitled *Methods of Computation* is devoted to means of computation of the single integral $\int_0^z e^{-x^2}dx$. Many known methods of evaluating this integral are given in the first few pages. These, in general, were developed for computation when z^2 was a real or pure imaginary number. The known methods are extended and new ones introduced for handling more general values of z. Ways of estimating the accuracy of the computation based on the various types is also given. Power series, asymptotic series, and continued fractions are the main devices used for making the evaluations. As by-products, methods for evaluating a great many other types of integrals are given; of these we mention only the Fresnel integrals.

Pages 102 to 179 are devoted to various means of evaluating the double integral mentioned in the title. Here, in general, the problem is reduced to evaluation of series and certain single integrals.

The last pages of the book are devoted to two tables. Let the functions $Rr(x)$ and $Ri(x)$ be defined by the equation

$$Rr(x) + iRi(x) = \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{e^{-\pi x^2 y^2}}{1 - iy^2}\,dy, \qquad i = \sqrt{-1}.$$

Table 1 gives the values of $Rr(x)$, $Ri(x)$, $Rr^2(x)+Ri^2(x)$, and $\int_0^z Rr(y)dy$ to 12 decimal places for the range $x = +.06, -.04, \cdots, 3.75$. Table 2 gives the values of

$$e^{-x^2}\int_0^x e^{y^2}dy, \qquad e^{x^2}\int_x^\infty e^{-y^2}dy, \qquad \int_0^x e^{-y^2}dy\int_0^y e^{t^2}dt, \qquad \int_0^x e^{y^2}dy\int_y^\infty e^{-t^2}dt$$

to 10 decimal places for the range $x = -.20, -.15, \cdots, 6$.

Considering the large number of formulas in the book there are relatively few misprints. Unfortunately, the format is not pleasing and there is little moti-

vation of the exposition. However, the book should prove of great value to those interested in computation of definite integrals.

F. G. DRESSEL

Modern-School Solid Geometry. By R. R. Smith and J. R. Clark. Yonkers-on-Hudson, N. Y. World Book Company, 1949. 8+256 pages. $1.76.

In the preface the authors state that the primary purposes to be achieved in the study of solid geometry are four-fold: (1) discovery of geometrical truths of three-dimensional space and their establishment by logical methods; (2) development of computational skill; (3) practice of perceptual analysis leading to the ability to see relationships in space; (4) extension of the understanding of the nature of logic.

This reviewer is in sympathy with this and believes that an able student, aided by a competent instructor, should certainly achieve the first three of the above purposes. It is questionable whether or not he will make much advance in the fourth.

The formal subject matter is contained in some twenty postulates, sixty-one theorems, and one hundred and twenty corollaries. There are nearly a thousand exercises most of which are numerical. For reference, there is a list of plane geometry theorems and constructions. In addition there are four place tables of trigonometric functions and common logarithms.

The format of the book is very pleasing. Diagrams and illustrations are excellent. It should be a very teachable book.

J. N. RICE

Kurvenintegrale and Begründung der Funktionentheorie. By Lothar Heffter. Berlin, Springer, 1948. 4+48 pp. DM 5.40.

This pamphlet is a unified treatment of seven papers of the author appearing between 1902 and 1941. The objective is to prove the Cauchy integral theorem: $\int_c f(z)dz=0$, under the weakest possible hypotheses. The classical proof assumes the *existence* and *continuity* of $f'(z)$ in the domain considered. As is well known, Goursat in 1900 weakened the hypotheses, assuming only the *existence* of $f'(z)$. The author avoids assuming the existence of $f'(z)$, replacing it by a difference equation.

The discussion is reduced to a consideration of real line integrals $\oint f(x, y)dx + g(x, y)dy$ which are in turn reduced to integrals around rectangles with sides parallel to the axes. The vanishing of the line integral around such a rectangle is then equivalent to an elementary difference equation.

The interest of the book lies with the following items:

1. The care and completeness with which the book is written. Chapters I through IV contain the requisite material on functions of one and two variables, making the work self-contained. The book is recommended to the student desiring to deepen his understanding of the Cauchy Theorem and at the same time improve his reading ability in German.

2. The author's proof of the Cauchy-Goursat theorem and his remarks concerning the excessive strength of the assumption that $f'(z)$ exists.

3. Chapter VII which consists of a bibliography (incomplete but containing the important items) and remarks on the contents of the papers listed.

The main contribution of the author could perhaps best be described as a new and easily understood proof of the Cauchy-Goursat Theorem. The fundamental difference equation is not, however, a local condition on $f(z)=u+iv$. On the other hand the Looman-Menchoff Theorem proves the Cauchy Theorem under essentially weaker hypotheses of a local nature (u and v merely possessing partial derivatives satisfying the Cauchy-Riemann equations at "most" points of the domain). The only mention of the Looman-Menchoff Theorem is in Chapter VII where the author remarks that its proof is quite complicated whereas his method leads to a simple proof.

M. E. SHANKS

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Mathematics Club, Haverford College

At the four meetings of the Haverford College *Mathematics Club* the following papers were presented:

The fourth dimension, by Irving Hollingshead

The cubic equation, by Charles Hughes

Hyperbolic trigonometry, by John Woll

Constructions: ruler and compass, by John Lamperti.

At each meeting prizes were awarded to members of the club for the best solutions to problems which had been posted previous to the meeting. The presentation of the solutions by the winners constituted a special feature of the club's activities.

Ricci Mathematics Academy, Boston College

The following lectures were presented to the members of the *Ricci Mathematics Academy* during the year:

A plane geometry which cannot be embedded in space, by F. A. Guerrera

The meaning of a mathematics academy, by Rev. A. J. Eiardi, S.J.

Elements of mathematical chance and probabilities, by P. T. Banks

Functions, functionals and operators, by Dr. H. G. Haefeli
Random samples, by Prof. Adelnour Thomas
Nomography, by Dr. F. E. White.

The second annual mid-term social and dance, sponsored by the Academy was held at the Princess Ballroom of the Hotel Somerset in Boston.

The officers of the Academy are: President, John Downes; Vice-President, Thomas Lynch; Secretary, William Durante; Treasurer, Raymond Walton; Program Committee, Julian Demeo, Francis Collins, Joseph Crosby, and Mark Devane.

Mathematics Club, Swarthmore College

The Swarthmore *Mathematics Club* had an active year during 1949–50. The following topics were presented by students, faculty members, and visiting professors:

Curves of unit diameter, by Louis Howard
Plane curves in projective differential geometry, by Prof. Alice Schafer
A locus problem, by Louis Winer
Certain problems in the foundations of geometry, by Prof. J. R. Kline of the University of Pennsylvania
Lattice points and number theory, by Prof. Hans Rademacher of the University of Pennsylvania
The calculus of finite differences, by Robin Briehl
Counting the infinite, by Prof. Philip Carruth
The story of numbers, by Mary Yntema
Complex and hypercomplex numbers, by Prof. Richard Schafer of the University of Pennsylvania
Some theorems on algebraic curves, by Prof. Arnold Dresden
Paper folding, by Robert Eisinger
An equation for Swarthmore, by Prof. C. O. Oakley of Haverford College
Symbolic logic, by Joseph Rutledge.

In addition to its regular meetings, the club sponsored two contests, each consisting of three problems of graded difficulty. A maximum mathematical preparation for each problem was specified as part of the eligibility requirements.

Officers for the year 1949–50 were: President, Robert Eisinger (fall) and Arthur Mattuck (spring); Treasurer, Mary Yntema.

The new officers elected are: President, Joan Berkowitz, Treasurer, Louis Winer.

Mathematics Club, Brown University

The *Mathematics Club* of Brown University, organized in 1915, heard the following papers during the year:

The probabilities in bridge, by Frank Stewart
The Königsberg bridge problem, by Margaret Hashimora
Oscillograph patterns, by George Anderson

Electronic computing machines, by E. F. Moore
Galois theory, by Philip Curtis
Pi, by Barbara Bruce
Mathematical tricks, by Robert Peabody
Continued fractions, by Joseph Crudele
Constructions by means of two parallel lines, by Gerald Freileich.

A winter social and a spring picnic were held. An informal period for discussion during which refreshments were served was held after each regular meeting.

The staff of the Mathematics Club were: Chairman, Alexander Marshall; Treasurer, Joseph Crudele; Secretary, Dorothy Smith; Committee members, Barbara Bruce, Philip Curtis, James Gray, Simone Matteodo, Robert Peabody, Chester Thomas, Barbara Webb: Faculty Representative, Prof. H. Federer.

Mathematics Club, Immaculate Heart College

The following papers were presented by student members of the club:
The humanistic side of mathematics, by Mary Parker
Approximate computation, by Marie Kreuper
History of calculus, by Rita Ricciardi
The life of Newton, by Helen Clark
The life of Leibnitz, by Glenna Christie
Playing with numbers, by Joan Pfisterer
Review of G. H. Hardy's A Mathematician's Apology, by Rita Anderson
Magic squares, by Mary Heffern
The slide rule, a motion picture.

At each meeting a period was devoted to mathematical recreations after which refreshments were served.

Officers for 1949–50 were: President, Mary Claire Heffern; Secretary-Treasurer, Glenna Christie.

Pi Mu Epsilon, University of Washington

The following papers were presented at regular meetings of the *Washington Beta* chapter of *Pi Mu Epsilon* during 1949–50:

Proof of the Weierstrass approximation theorem, by Dr. Edwin Hewitt
Proof that e is a transcendental number (a modification of Edourd Maillet's proof), by Halton McQueen
The fifth postulate and some consequences of its omission, by Calvin Thimsen
Spherical conics, by Dr. Mo-Yeh
Statistical aspects of the Geiger counter, by Richard Titman
Some properties of continued fractions, by C. Y. Lee
Evaluating an infinite product, by Richard Woollett
Application of calculus in high school physics, by Orlando Furno.

The regular officers for the 1950–51 will be elected in the Fall. However, Richard Woollett has been elected temporary Director and Orlando Furno has been elected temporary Secretary-Treasurer.

Kappa Mu Epsilon, The College of Wooster

During the academic year 1949–50, the following papers were presented at meetings of the *Ohio Beta* chapter of *Kappa Mu Epsilon*:

Mathematics and philosophy, by Prof. Wilford Bower

The calculus of variations, by Prof. M. P. Fobes

Trisection of the angle, by Charles Sauder

The cooperative engineering plan of the University of Cincinnati, by Prof. Emeritus J. H. Kindle of the University of Cincinnati.

The program also included a picnic at the home of Prof. C. O. Williamson, two initiations and social meetings.

Officers for 1950–51 are: President, Richard Holroyd; Vice-President, Morley Russell; Secretary, Milton Snyder; Treasurer, James Kister; Sponsor, Prof. M. P. Fobes.

Mathematics Club, Albany State College for Teachers

The program of the *Mathematics Club* at the New York State College for Teachers at Albany was composed of talks by students, a panel discussion led by students, a trip to the plant of General Electric at Schenectady, and a report of a professional meeting by a faculty member.

The following talks were given:

The Moebius strip, by Wayne Palmer

What other mathematics clubs have done, by Virginia Roth

Celestial navigation, by Jack Mooney

The unit circle, by Virginia McDonald

The binomial theorem, by Ifigenia Aliferis.

The panel discussion on high school mathematics clubs was conducted by Shirley Casler and Henrietta Daub, a discussion that was prompted by a desire to learn how to organize and develop a mathematics club in high school. The discussion, which involved an interesting piece of research, provoked considerable interest.

The trip to the General Electric plant was made to see the differential analyzer in operation and to learn something about how it worked. Dr. Caroline Lester gave the report of the Christmas meeting in New York.

The officers for the year were: President, Ruth Marschner, Secretary, Marjorie Hills; Treasurer, Rosemary Lessard; Faculty Adviser, Nura Turner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

VACANCIES AT U. S. NAVAL ORDNANCE TEST STATION

The United States Naval Ordnance Test Station, Inyokern, China Lake, California, is recruiting a large number of technical personnel as a result of the present world situation. The Station is establishing a group of senior mathematicians and statisticians for Weapons Evaluation work. It is also seeking theoretical physicists with specialized interests in Solid State Physics, Atomic Physics, Fluid Mechanics, Hydro or Aero Dynamics, and in almost any field of Classical Physics. Vacancies exist also in most fields of Engineering. There are openings for recent graduates with degrees ranging from Bachelor of Arts to Doctor of Philosophy in the above fields.

Full information and application forms can be obtained from: J. B. Hamilton, Head Professional Placement Branch, U.S.N.O.T.S., Inyokern, China Lake, California.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service is offering for 1951–52 its fourth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,375 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a pre-requisite for obtaining these fellowships. Applications must be submitted before January 19, 1951. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

GRADUATE INSTITUTE FOR APPLIED MATHEMATICS AT INDIANA UNIVERSITY

Indiana University has recently established a Graduate Institute for Applied Mathematics. The work of the Graduate Institute will deal primarily at present with problems in gas dynamics, hydrodynamics and turbulence. Members of the Graduate Institute will operate under exceptionally favorable teaching arrangements to allow ample time for their own researches and for personal contacts with students. It is advisable for students entering the Graduate Institute to have the M.A. degree and at least one year of graduate work in mathematics is required. A number of research assistantships open to students who wish to

work toward the Ph.D. degree in applied mathematics have been established by Indiana University under contract with the Office of Naval Research.

CONFERENCES

The second annual Conference of Teachers of Mathematics was held at Marshall College, Huntington, West Virginia, on March 24–25, 1950. At this conference the teachers of college mathematics of West Virginia organized the Association of College Mathematics Teachers of West Virginia. The following officers were elected: President, Professor A. V. Kozak of Concord State College; Secretary-Treasurer, Professor J. K. Sterrett, Marshall College.

On March 18, 1950 Millersville State Teachers College, Millersville, Pennsylvania, held a High School Mathematics Conference. Another conference is scheduled for March 10, 1951.

PERSONAL ITEMS

An honorary degree of Doctor of Laws was conferred upon Dean E. A. Bailey by Florida Southern College on March 17, 1950 for his twenty-five years of service as Dean of LaGrange College.

Arizona State College, Flagstaff, reports the following: Dr. Arthur Adel was an invited speaker at the Centenary Meeting of the Royal Meteorological Society which was held on March 27–April 4, 1950 at Oxford and London, England; Professor J. H. Butchart was the travelling lecturer for the Southwestern Section of the Association for the year 1949–50.

Arizona State College, Tempe, announces: Associate Professor F. C. Gentry has been promoted to a professorship; Assistant Professor R. S. Fouch of the University of Chicago has been appointed to an assistant professorship.

Baldwin-Wallace College announces the following: Associate Professor P. R. Annear, head of the Department of Mathematics and Astronomy, has been promoted to a professorship; Assistant Professor D. L. Robb has been promoted to an associate professorship in the Department of Mathematics and Astronomy.

Fullerton Junior College, California, reports: Mr. C. A. Worsley, chairman of the division of Mathematics, Science and Engineering, has retired; Mr. L. W. Wheatley has succeeded him.

Indiana University makes the following announcements: Professor T. Y. Thomas, Professor Eberhard Hopf and Assistant Professor David Gilbarg have been transferred from the Mathematics Department to the newly established Graduate Institute for Applied Mathematics; Dr. C. A. Truesdell of the Naval Research Laboratory, Washington, D. C., has been appointed to a visiting professorship for the year 1950–51.

At Montana State University: Dr. George Marsaglia, previously graduate student at Ohio State University, has been appointed to an assistant professorship; Mr. G. A. Craft, graduate student at the University of Indiana, and Mr. A. L. Duquette, graduate student at Columbia University, have been appointed to instructorships; Dr. H. B. Mann of Ohio State University was a

visiting lecturer during the 1950 Summer Session.

North Dakota Agricultural College announces the following: Instructor G. A. Walrath has been promoted to an assistant professorship; Mr. David Gosslee has been appointed to an instructorship.

North Georgia College reports the following: Professor J. C. Barnes, head of the Department of Mathematics, has retired; Dr. M. C. Wicht of Louisiana State University has been appointed Professor and Head of the Department of Mathematics; Mr. B. F. Hadnot, graduate student at the University of Georgia, has been appointed to an instructorship; Associate Professor G. N. Wollan is engaged in graduate study at the University of Georgia.

Pomona College reports: Professor C. G. Jaeger has been elected Governor of the Southern California Section of the Association; Mr. C. J. A. Halberg, Jr. has left Pomona College and is doing graduate work at the University of California at Los Angeles.

Trinity College announces the appointments of Mr. R. C. Stewart and Mr. J. N. Williams to instructorships.

The United States Naval Postgraduate School announces the following promotions: Associate Professors W. E. Bleick and C. C. Torrance to professorships; Assistant Professor Walter Jennings to an associate professorship.

At University of Bridgeport: Associate Professor Simon Mowshowitz has been promoted to the position of Professor and Chairman of the Department of Mathematics; Professor E. M. Bigsbee, formerly chairman of the Department of Mathematics, has been appointed Dean of the Junior College of Connecticut.

University of California, Santa Barbara College, reports the following: Associate Professor S. E. Rauch, chairman of the Department of Mathematics, is on sabbatical leave during the year 1950–51 and is studying in France; Assistant Professor L. F. Walton is Acting Chairman of the Department for the year.

University of Washington makes the following announcements: Associate Professor Z. W. Birnbaum has been promoted to a professorship; Assistant Professor Edwin Hewitt has been promoted to an associate professorship; Professor Kôsaku Yosida of the Mathematical Institute of Nagoya University has been appointed Walker-Ames Professor for the autumn quarter; Dr. F. H. Brownell III of Princeton University and Dr. R. B. Leipnik of the Institute for Advanced Study have been appointed to assistant professorships; Dr. R. P. Peterson of the University of California at Los Angeles has been appointed to an instructorship; Assistant Professor Edward Paulson has returned from a leave of absence spent at Columbia University; Lecturer D. M. Sandelius has returned to Sweden; Dr. Yeh Mo has returned to China.

At University of Western Ontario: Professor H. R. Kingston has resigned from the position of Head of the Department of Mathematics and Astronomy to accept the position of Principal of University College and Dean of Arts and Science; Associate Professor G. R. Magee has been promoted to a professorship and named Head of the Department of Mathematics and Astronomy; Associate Professor R. H. Cole has been promoted to a professorship; Dr. G. P. Henderson has been promoted to the position of Lecturer.

Dr. Shmuel Agmon has been appointed Visiting Lecturer in Mathematics at the Rice Institute.

Dr. Gerald Berman has been appointed to an instructorship at Illinois Institute of Technology.

Assistant Professor F. C. Bolser of Florida State University has accepted an assistant professorship at the University of Tampa.

Mr. C. B. Eaton has been appointed to an instructorship at Pasadena City College.

Associate Professor C. H. Fischer of the University of Michigan has been promoted to a professorship of actuarial mathematics in the College of Literature, Science and the Arts and has also been appointed Professor of Insurance in the School of Business Administration.

Professor J. S. Frame, head of the Department of Mathematics of Michigan State College, is spending the year 1950–51 at the Institute for Advanced Study.

Instructor Hunter Hardman of Marshall College has been promoted to an assistant professorship.

Assistant Professor C. B. Helms of Pennsylvania Military College has been promoted to an associate professorship.

Assistant Professor P. B. Johnson has been appointed Acting Chairman of the Department of Mathematics of Occidental College and promoted to an associate professorship.

Assistant Professor R. E. Lowney has been promoted to an associate professorship at Montana State College.

Mr. R. C. Lundquist has been appointed to an instructorship at State Teachers College, Frostburg, Maryland.

Assistant Professor Walter Lyche of Augustana College, Sioux Falls, South Dakota, is now a graduate student at the University of Minnesota.

Assistant Professor H. C. McKenzie of Western State College of Colorado has been appointed to an instructorship at the University of Colorado.

Mr. R. B. Merrill has accepted a position as statistician in the Quality Control Department of Wisconsin Steel Works, Chicago, Illinois.

Associate Professor N. H. Mewaldt, Northern State Teachers College, Aberdeen, South Dakota, has been elected to membership in the Beadle Club, an honorary educational society in South Dakota.

Dean E. F. Moore of Hannibal-LaGrange College has been appointed Professor of Statistics at Baylor University.

Dr. H. B. Ribeiro of the University of California at Berkeley has been appointed to an associate professorship at the University of Nebraska.

Associate Professor W. H. Spragens, Jr. of Florida State University has been appointed to an associate professorship at the University of Mississippi.

Professor I. L. Stright of State Teachers College, Indiana, Pennsylvania, is on leave of absence for the year 1950–51 and is teaching at the Higher Teachers Training College, Baghdad, Iraq.

Miss Mabel G. Whiting, head of the Department of Mathematics of Santa Ana Junior College, California, has retired.

Professor Henry Blumberg of Ohio State University died on June 28, 1950. He was a charter member of the Association.

Professor W. J. Bruns of Utica College, Syracuse University, died July 18, 1950.

Assistant Professor R. I. Pepper, acting head of the Department of Mathematics of Winthrop College, died July 2, 1950.

Miss Elinor Schneider, instructor at Pasadena City College, died on January 1, 1950.

Professor Emeritus E. W. Sheldon of Acadia University died June 14, 1950. He was a charter member of the Association.

Professor W. M. Smith, head of the Department of Mathematics of Lafayette College and a charter member of the Association, died on July 3, 1950.

Professor G. R. Trott of the University of Mississippi died on July 8, 1950. He was Chairman of the Louisiana-Mississippi Section for the year 1949–50

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and one persons have been elected to membership by the Board of Governors on applications duly certified.

W. A. BALLOU, Ph.D. (Pennsylvania) Professor, Morehead State College, Ky.

E. H. BATHO, Student, Fordham University, New York, N. Y.

J. R. BATT, Student, University of Buffalo, N. Y.

S. K. BERBERIAN, M.S. (Michigan S.) Grad. Student, Michigan State College, East Lansing, Mich.

H. G. BERGMANN, Ph.D. (N.Y.U.) Instructor, City College of the City of New York, N. Y.

JEROME BERKOWITZ, B.S. (C.C.N.Y.) Research Assistant, New York University, N. Y.

R. J. BICKEL, M.A. (Northwestern) Asst. Professor, Drexel Institute of Technology, Philadelphia, Pa.

H. F. BLIGH, M.S. (Syracuse) Asst. Professor, Hobart College, Geneva, N. Y.

C. R. BONNELL, B.S. (Central Missouri C.) Instructor, Missouri School of Mines & Metallurgy, Rolla, Mo.

J. C. BRADFORD, Student, North Texas State College, Denton, Texas

C. L. BRADSHAW, B.S. (Tennessee Poly. Inst.) Grad. Assistant, University of Tennessee, Knoxville, Tenn.

MRS. HELEN E. BROWN, A.M. (Michigan) Instructor, University of Tennessee, Knoxville, Tenn.

C. C. BUCK, M.S. (Michigan) Instructor, University of Nebraska, Lincoln, Nebr.

M. D. BURACK, M.S. (DePaul) Instructor, Wilson Junior College, Chicago, Ill.

SARAH M. BURKHART, M.A. (Kansas) Asst. Professor, University of Tulsa, Okla.

B. A. BURRILL, Student, University of Buffalo, N. Y.

G. J. CAMPBELL, Student, University of Detroit, Mich.

R. S. CHRISTIAN, M.S. (Washington U.) Asst. Professor, Georgia Institute of Technology, Atlanta, Ga.

S. I. CIOLKOWSKI, Student, New York University, N. Y.

J. P. CORY, Student, New York University, N. Y.

Q. A. DARMSTADT, Student, Oberlin College, Ohio

D. J. DAVIS, B.S. (Harvard) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

RUTH M. DAVIS, Student, American University, Washington, D. C.

ANNE F. DOWNEY, Student, Regis College, Weston, Mass.

R. H. EDWARDS, Ph.D. (Illinois Inst. of Tech.) Research Physicist, Hughes Aircraft Co., Culver City, Calif.

J. O. ETHRIDGE, B.A. (Southwestern at Memphis) Grad. Assistant, University of Georgia, Athens, Ga.

T. C. FRANCIS, Accountant, Hanley & Bird, Bradford, Pa.

LOUIS GATT, Student, University of California at Los Angeles, Calif.

R. H. GILLESPIE, Student, Albion College, Mich.

N. A. GOLDSMITH, M.A. (Illinois) Asso. Professor, Illinois Wesleyan University, Bloomington, Ill.

H. H. GOODE, M.A. (Columbia) Supervisor, Aerophysics Group, Aeronautical Research Center, University of Michigan, Ann Arbor, Mich.

E. E. GREEN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

R. E. GREEN, Student, Kansas State Teachers College, Pittsburg, Kan.

B. W. GREGORY, Student, Oregon State College, Corvallis, Ore.

J. C. HARDEN, JR., M.A. (Tennessee) Instructor, Clemson College, S. C.

J. J. HART, B.S. (Memphis S. C.) Grad. Student, University of Alabama, University, Ala.

R. F. HAYS, A.B. (Transylvania) Grad. Student, University of Kentucky, Lexington, Ky.

P. G. HODGE, JR., Ph.D. (Brown) Asst. Professor, University of California at Los Angeles, Calif.

SZE-TSEN HU, Ph.D. (Manchester, England) Asst. Professor, Tulane University, New Orleans, La.

H. A. JAMES, M.S. (DePaul) Instructor, Wilson Junior College, Chicago, Ill.

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SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Association was held at the University of Richmond in Richmond, Virginia, on Saturday, May 6, 1950. Professor L. M. Kells, Chairman of the Section, presided at both the morning and afternoon sessions.

Fifty-seven people registered attendance at the meeting including the following forty-three members of the Association: M. W. Aylor, R. P. Bailey, W. E. Barnes, R. H. Bing, J. W. Blincoe, Eleanor C. Calkins, Randolph Church, G. R. Clements, C. H. Frick, B. C. Getchell, M. V. Gibbons, Michael Goldberg, R. A. Good, E. S. Grable, Clem Grabner, Jr., E. C. Gras, D. W. Hall, J. R. Hammond, Isabel Harris, J. E. Ikenberry, S. B. Jackson, Walter Jennings, Sidney Kaplan, L. M. Kells, V. L. Klee, Jr., D. B. Lowdenslager, Joseph Milkman, J. F. Milos, A. K. Mitchell, Dewey Moore, Mary Pettus, F. M. Pulliam, O. N. Robinson, R. C. Simpson, Jr., T. McN. Simpson, Jr., E. R. Sleight, Robert Edward Smith, John Tyler, C. H. Wheeler, III, P. M. Whitman, G. T. Whyburn, Verba M. Wood, J. W. Wright.

The following officers of the Section were elected for the year 1950–51: Chairman, O. J. Ramler, Catholic University; Secretary, S. B. Jackson, University of Maryland; Executive Committee, E. R. Sleight, University of Richmond, and R. P. Bailey, U. S. Naval Academy. Professor Kells announced that the Bureau of Standards in Washington, D. C. has invited the Section to meet there for one of the meetings in 1951. He also announced that Professor G. R. Clements of the Naval Academy has been elected Regional Governor from this Section for the next three years. The next meeting of the Section will be in December 1950 at a place to be announced later.

The morning session consisted of five contributed papers, as follows:

1. *The constants of integration of a differential equation as differential forms*, by Professor John Tyler, United States Naval Academy.

If the equation $y=f(x, c_0, c_1, \cdots, C_K)$ is differentiated K times, the resulting $K+1$ equations may be solved for the C_i as functions of x, y, and the first K derivatives of y. Thus the c_i are expressed as differential forms in x, y, and its derivatives, namely, $c_i=c_i(x, y, y', \cdots y^{(K)})$, $i=0, 1, \cdots, K$. These differential forms have certain interesting properties, notably that $dC_i=\lambda_i D dx$ where $D=0$ is a differential equation of order $K+1$ satisfied by the function f, and the λ_i are integrating factors. If the C_i are substituted into the equation $F(C_0, C_i, \cdots C_K)=C$, where F is an arbitrary function, there arises an equation of Clairaut type for which the function f is a solution.

2. *Some characterizations of compactness*, by Professor V. L. Klee, Jr., University of Virginia.

Niemytzke and Tychonoff showed by an adaptation of the Alexandroff-Urysohn metrization proof that a metrizable space is compact if and only if it is complete in each of its allowable metrics. Hausdorff proved that each allowable metric on a closed subset of a metric space can be extended to an allowable metric on the whole space. By use of the fact, he gave a simple proof of the above result, and showed that "complete" can be replaced by "bounded." Vaughan improved on these characterizations by further use of Hausdorff's extension theorem. In this note there is established an elementary embedding theorem which is then applied to give rather simple and pictorial proofs

for Vaughan's principal results as well as to establish two new characterizations of compactness which seem not to be directly obtainable from the extension theorem.

3. *Rotors in polyhedra*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

A rotor in a polyhedron was defined as a closed convex surface which can be rotated continuously through every orientation within a given polyhedron while keeping contact with all the faces of the polyhedron. Rotors in cubes, which are rotors in other rhombic parallelepipeds also, have been studied by Minkowski. By elementary methods, Mr. Goldberg derived a class of non-spherical rotors for the regular tetrahedron and exhibited working models of them. An outline was presented of E. Meissner's derivation of all the possible rotors for the regular polyhedra. For the dodecahedron and the icosahedron, the sphere is the only possible rotor.

4. *On finding* $\sqrt{a+bi}$, by Mr. Sidney Kaplan, U. S. Department of Labor.

The standard methods for computing $\sqrt{a+bi}$, namely De Moivre's theorem and the solution of a system of quadratics, both present difficulties for machine computation, the first because of the number of steps, the second because of the loss of significant figures. A slight modification of the second method, suggested by Professor Garrett Birkhoff, proves to be both accuracy preserving and labor saving.

5. *Undergraduate research*, by Professor E. R. Sleight, University of Richmond.

This paper dealt with the need and value of a program that will challenge the student of unusual ability. Professor Sleight feels that some form of undergraduate research has created a great deal of interest, and is already operating in many institutions. He discussed the introduction and maintenance of this type of work in various institutions with which he has had acquaintance.

6. *Some applications of partitionings*, by Professor R. H. Bing, University of Wisconsin and University of Virginia.

This was an invited address. Suppose X is a compact locally connected metric continuum. An ϵ-partitioning of X is a finite ϵ-collection $G=(g_1, g_2, \cdots, g_n)$ of mutually exclusive open subsets of X such that $\sum g_i$ is dense in X. If $G_1, G_2, \cdots$ is a sequence of partitionings, it is called a decreasing sequence if G_{i+1} refines G_i and the diameters of the elements of G_i approach 0 with $1/i$. The following applications of the fact that X has a decreasing sequence of partitionings were discussed. There is a convex metric for X. Each pair of points of X belongs to an arc whose complementary domains in X have property S. There is a countable basis for X such that the intersections of the elements of this basis are connected and uniformly locally connected. The notion of partitionings may be used to characterize simple closed curves, simple surfaces, and simple solids.

S. B. JACKSON, *Secretary*

APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Syracuse University, Syracuse, New York, on Saturday, April 22, 1950. The Chairman of the Section, Professor Walter H. Durfee of Hobart and William Smith Colleges, presided at the morning session; the Vice-Chairman, Professor B. C. Patterson of Hamilton College, presided at the afternoon session. Tea was served at the conclusion of the afternoon session.

One hundred twenty-five persons attended the meeting including the follow-

ing seventy-four members of the association: E. B. Allen, H. P. Atkins, D. C. Barton, M. R. Bates, R. W. Beals, Jr., R. L. Beinert, Dorothy L. Bernstein, W. W. Bessell, Jr., C. G. Burger, Jr., F. J. H. Burkett, E. A. Butler, Ethel B. Callahan, J. D. Campbell, I. S. Carroll, W. B. Carver, P. F. Clemens, Nancy Cole, E. J. Downie, Walter H. Durfee, Jean B. Feidner, C. W. Foard, A. H. Fox, J. E. Freund, H. M. Gehman, B. H. Gere, H. E. Goheen, R. D. Gordon, Lillian Gough, J. B. Greeley, N. G. Gunderson, May N. Harwood, M. B. Haslam, Lucille F. Hetzelt, R. G. Hill, H. K. Holt, A. W. Jones, J. R. F. Kent, D. E. Kibbey, R. D. Larsson, J. V. Limpert, R. R. R. Luckey, June M. McArtney, L. L. Merrill, Rudolf Meyer, F. E. Milliman, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, Abigail M. Mosey, C. W. Munshower, C. V. Newsom, Ruth B. Noller, B. C. Patterson, Ruth M. Peters, Theresa L. Podmele, M. H. Protter, J. F. Randolph, H. W. Reddick, C. E. Rhodes, M. K. Rosskopf, Edith R. Schneckenburger, Joseph Seidlin, Sister Noel Marie, William A. Smith, Ruth W. Stokes, Mary C. Suffa, G. L. Tiller, J. P. van Alstyne, S. I. Vrooman, R. J. Walker, J. F. Wardwell, E. T. Welmers, Ina W. Welmers, R. C. Yates.

A progress report was received from the committee studying the relation between secondary school and college mathematics in New York State, Professor M. F. Rosskopf, Syracuse University, Chairman.

The following officers were elected: Chairman, B. C. Patterson, Hamilton College; Vice-Chairman, C. W. Munshower, Colgate University; Secretary, N. G. Gunderson, University of Rochester. The 1951 meeting will be held at Hamilton College, Clinton, New York, and the 1952 meeting at Hobart and William Smith Colleges, Geneva, New York.

The following papers were presented:

1. *Models and methods*, by Professor R. C. Yates, U. S. Military Academy.

A few items intended to stimulate student interest in both subject matter and technique were presented. These included folding the conics, the current misuse of Euclid's compass, dissection of plane polygons, and simple derivations of normal acceleration, the Euler exponential form, and the probability integral. Several models were exhibited as suggested aids to the teacher.

2. *The problem of eight points*, by Professor W. B. Carver, Cornell University.

This paper has been published in this MONTHLY, May, 1950.

3. *Some educational problems in New York State*, by Dr. C. V. Newsom, Assistant Commissioner for Higher Education, New York State.

This speaker gave a brief discussion of the state educational program. In particular, emphasis was given to the role that mathematics departments might play in the training of elementary and secondary teachers. Comments were also made upon the inferior level of college instruction in many institutions.

4. *On space frames and plane trusses*, by Professor H. E. Goheen, Syracuse University.

The speaker proposed the following definition for use by the student of elementary mechanics: A space frame (or plane truss) will be called statically determinant if and only if the $3j$ (or $2j$)

conditions for equilibrium at the j joints determine uniquely the m bar forces and the 6 (or 3) forces necessary to restrain the structure.

5. *Analytic solutions of non-linear differential equations*, by R. D. Larsson, Clarkson College of Technology.

The purpose of this paper is to present a general method for solving non-linear differential equations with polynomial coefficients. The method of Laplace transforms has been used to achieve an analytical solution. Some equations of this type have been solved by the method of approximations; but the method presented here gives the solution in a form in which the role of the boundary values is apparent for any particular case. The general solution reduces to known results in special cases. For the purpose of illustration a second order differential equation with constant coefficients, non-linear in the first derivate, is used.

6. *Difference and differential equations in probability theory*, by Dr. Murray Rosenblatt, Cornell University, introduced by Professor R. J. Walker.

Models of 1 and 2-dimensional Brownian motion and corresponding 1 and 2-dimensional random walks are defined. The probability density of the particle in Brownian motion is a fundamental solution of a parabolic differential equation while the probability that the particle in the random walk be at a lattice point is a solution of a corresponding difference equation. Absorption problems for Brownian motion and the random walk are discussed and related to boundary value problems of the corresponding differential and difference equations. This is made plausible by considering in some detail the 1-dimensional Brownian and random walk. The probability distribution of the Brownian motion is a limit of the solution of the random walk problem.

7. *The tutorial system at the University of Buffalo*, by Professor Harriet F. Montague.

The tutorial system at the University of Buffalo provides that all students in their last two years of undergraduate work shall be given an opportunity for independent study and investigation in addition to regular course work. The tutorial program of the department of mathematics is divided into three parts. In the junior year the students meet in small groups to study and discuss the history and foundations of mathematics, and its various branches. The first half of the senior year is devoted to independent work on an individual research problem. The last half of the senior year is concerned with a general review and correlation of all courses taken in mathematics. This is climaxed by a comprehensive examination.

C. W. MUNSHOWER, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The thirty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State Teachers College, Pittsburg, Kansas, on Saturday, April 22, 1950. Sessions were held in the morning and afternoon. Professor R. G. Smith presided at these sessions. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

The attendance was one hundred sixty-seven including the following fifty members of the Association: R. W. Babcock, C. H. Brown, W. R. Cowell, L. E. Curfman, Paul Eberhardt, W. C. Foreman, Albert Furman, W. H. Garrett, F. C. German, Laura Z. Greene, B. H. Gundlach, J. R. Hanna, N. C. Hoover, S. L. Hull, Emma Hyde, Edward Keegan, Helen F. Kriegsman, L. E. Laird,

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The program consisted of the following papers:

1. *Mathematics in general education*, by Mr. H. W. Charlesworth, East High School, Denver, Colorado.

This paper was given in the interest of better mathematics teaching in both secondary schools and colleges. Mathematics must make its maximum contribution to all students, recognizing their individual interests and needs. College instruction must give more attention to the art of teaching. Mathematics education for the prospective teacher of mathematics must be sound mathematically and educationally. Teacher training schools have a great responsibility in properly training teachers for this new era in mathematics. Mathematics can make its proper contribution to the general education of the individual and yet retain its value as a mode of thought. The training of our leaders in mathematics and science must begin early and be done more expertly. Greater coordination of high school and college teaching is needed. The Mathematical Association of America and The National Council of Teachers of Mathematics should do more cooperative work on solving their common problems arising out of general education.

2. *Sign lines and tangent carriers, a speedy check for curve tracing*, by Miss Henrietta Courtright, Arkansas City Junior College, introduced by the Secretary.

Since any function or factor occurring to the first power divides the plane into a negative and a positive region, this fact may be used as a rapid check for curve tracing. A square of any factor does not divide the plane into two regions but instead carries the point of tangency with a linear factor on the other side of the equation. Nodes and isolated points may be checked by looking for factors to the second power on each side of the equation, or a cusp may be checked in like manner when a factor to the second power is paired with a factor to the third power.

3. *Linear transformations of point lattices*, by Mr. W. R. Cowell, Kansas State College.

A unimodular transformation defines a mapping of a point lattice onto itself. It is possible to establish conditions for one lattice point to be carried into another under such a transformation, and also conditions for two corresponding pairs of lattice points to determine uniquely a unimodular transformation. This type of transformation preserves any order of collinear lattice points and also leaves invariant the number of lattice points on a closed convex plane curve and in the interior of the curve. Associated with a unimodular transformation of determinant $+1$ is an invariant binary quadratic form. This makes possible the use of the transformation as a recursion relation for the solutions of certain Diophantine equations.

4. *Selected applications in difference equations*, by Prof. J. R. Hanna, Wichita University.

This speaker introduced some of the notation, definitions, and procedures pertaining to difference equations. Consideration was given to the use of difference equations in solving problems relative to geometry, continued fractions, Fibonacci's numbers, and mathematics of investment.

5. *A philosophic synthesis of the sciences*, by Sister M. Helen Sullivan, Mount St. Scholastica.

This paper is an epistemological discussion of the findings of science, and shows the place and function of these findings in the general hierarchy of learning. Speculative sciences are classified according to the three degrees of formal abstraction. The specific role of each science as well as its relationship to other branches of knowledge is clearly brought out. The philosophic synthesis is established when a comparative study is made of all the sciences in the light of ultimate explanations. The synthesis is confirmed when science is regarded as one among several rational approaches to truth rather than as a mere technique.

6. *Bessel functions*, by Mr. L. E. Trapp, Kansas State College, introduced by the Secretary.

This paper is concerned with some of the aspects of the theory and application of Bessel functions. The theory involves the discussion of such topics as a second solution to Bessel's differential equation, zeros of $J_0(x)$, solution of the general equation, differentiation and recursion formulas, integral forms of $J_n(x)$, and orthogonality of Bessel functions. The application involves Bessel functions as applied to the solution of Kepler's problem on planetary motion.

ANNA MARM, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, December, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, Duluth Branch of University of of Minnesota, October 7, 1950.

MISSOURI, CENTRAL College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, April 21, 1951.

OKLAHOMA, Oklahoma City, November 14, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

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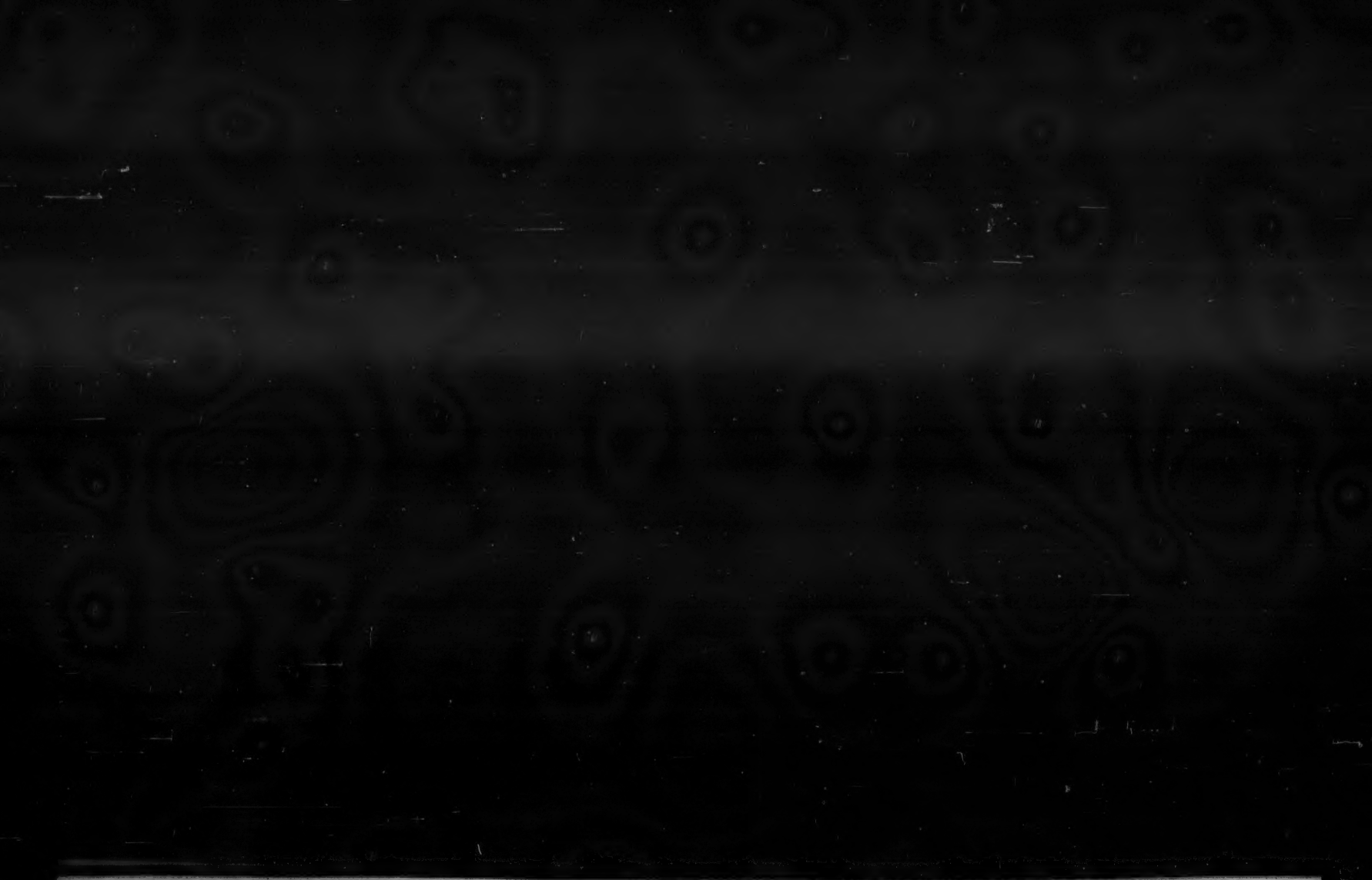

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